14. Solonnikov, V.A., A priori estimates for the second order parabolic equations. Tr. Mat. Inst. im. V.A. Steklova, Vol. 70, 1964.

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ON THE KINEMATICS, NONEQUILIBRIUM THERMODYNAMICS, AND RHEOLOGICAL RELATIONSHIPS IN THE NONLINEAR THEORY OF VISCOELASTICITY

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Within the scope of the customary thermodynamics of irreversible processes (TIP) (a linear connection between thermodynamic fluxes and forces, symmetry of the kinetic coefficients), and utilizing the relationship derived herein between reversible, irreversible, and total strain rates, a system of governing equations is constructed for the simplest viscoelastic media in the presence of arbitrary finite reversible deformations.

These equations are investigated in the case of sufficiently small reversible deformations; a "second-order" theory is constructed taking into account the physical as well as the geometrical, nonlinearity in the system. It is hence taken into account that the kinetic coefficients will be tensor functions of the tensor of reversible deformations. This latter leads to "deformation anisotropy" of the heat conduction and diffusion. Expressions are written down for entropy production in the system for the simplest model media.

The "second-order" theory is extended to the case of isothermal deformation of viscoelastic media with many relaxation times. The solution of a number of problems for the simplest flows (simple shear, tension) of viscoelastic media showed a good enough qualitative agreement between the constructed theory and experiment. Also questions about the inversion of the Jaumann tensor derivative ("Jaumann integration") are considered.

A large quantity of papers (see the survey [1]) is devoted to a theoretical description of viscoelastic media. In the phenomenological construction of a theory of viscoelasticity, as in the construction of continuum models generally [2 and 3], invariance considerations, the geometry of finite deformations, and thermodynamics are utilized, while the thermodynamics of irreversible processes (TIP) is used for dissipative media. Biot [4 and 5] made a sufficiently complete investigation of linear viscoelasticity under conditions of small velocities of this kind.

Let us refer to the work of Kluitenberg in which the thermodynamic derivation of governing equations for various media is expounded [6 to 9].

Among the earliest investigations on the nonlinear theory of viscoelasticity is the paper [10]; however, the kinematics of viscoelastic phenomena remained unclarified in this work, and there is a total absence of a thermodynamic analysis of the phenomena.

The development of a theory of nonlinear behavior of dissipative media is often connected with the extension of TIP [11]. In opposition to such a viewpoint, an attempt is made herein to utilize the customary version of TIP with linear phenomenological laws and Onsager reciprocity relationships, to derive the governing equations of a nonlinear viscoelastic medium with physical and geometric nonlinearities.

We shall often rely on [2 and 12] without detailed referral in expounding the theory of deformation of dissipative media and TIP.

It is known that assignment of a state function of internal-energy or entropy types (or of other thermodynamic potentials), which depend on the temperature and external parameters, is fundamental for the thermodynamical equilibrium processes. For small deviations from *equilibrium ("slightly dissipative" media) it is possible to assume conservation of such a* description with the aid of the state function.

First, it is generally necessary to increase the quantity of governing parameters (for example, to include some internal parameters among the arguments of the state function); secondly, it is necessary to give, in addition, the dissipative function which describes entropy production in a thermodynamic system.

The specific internal energy is selected as the state function; and it is assumed that it depends only on the specific entropy s and the reversible part of the deformation $\varepsilon_{ij}{}^{e}$ without additional internal parameters, i.e., the dependence $u(s,\varepsilon)$ is similar to that which holds in a nondissipative elastic medium. Only the lowest terms in the deviation from equilibrium are kept in the expression for the dissipation.

Such a thermodynamic consideration of a viscoelastic medium has analogy with the statistical approach to its hydrodynamics, when the description using a local equilibrium distribution is selected as the original distribution, and relaxation processes are taken into account as small deviations from this equilibrium distribution [13]. Let us note that the assumptions made essentially differentiate the viscoelastic medium under consideration from a medium with plastic deformations since the characteristic peculiarity of this latter is the dependence of the internal energy on at least the irreversible component of the deformation as well [6 and 9].

1. Kinematics of finite deformations in a viscoelastic medium. *Following [2], let us determine the reversible deformation in a medium particle by using some imagined, or actually producible process of unloading from stresses of a small particle.*

Let us define the unloading process of the given particle of the medium as its being released instantaneously from stresses and waiting during an infinite time interval. If the total deformation in a particle is $\varepsilon_{ij}(t_0)$ at time t_0, then at $t_0 + 0$ it changes by an "instantaneous" *elastic component, and furthermore, for $t > t_0$ it will be released from "delayed"* elastic deformation, so that only one irreversible deformation component $\varepsilon_{ij}{}^{p}$ remains in the particle as $t \to \infty$. The difference $\varepsilon_{ij}{}^{p} - \varepsilon_{ij} = \varepsilon_{ij}{}^{e}$ defines the reversible component of the deformation. The quantity $\varepsilon_{ij}{}^{e}$ is determined experimentally in precisely this fashion (with the sole exception that the test lasts a finite time).

Let us introduce a Lagrangean "frozen" coordinate system ξ^1, ξ^2, ξ^3 and let us consider three positions of the continuum relative to a fixed x^1, x^2, x^3 coordinate system with the vector basis ϑ^i and the fundamental form

$$ds^2 = g_{ij}dx^i dx^j$$

1) The initial position at time $t_0 < t$ with basis $\vartheta_0{}^i$, fundamental form $ds_0{}^2 = g_{ij}{}^{(0)} d\xi^i d\xi^j$;

2) The deformed state at time t with basis $\vartheta_1{}^i$, fundamental form $ds_1{}^2 = g_{ij}{}^{(1)}(\xi^k, t) d\xi^i d\xi^j$;

3) The "unloading" state at time $t + \infty$ with basis $\vartheta_2{}^i$ and fundamental form $ds_2{}^2 = g_{ij}{}^{(2)}(\xi^k, t + \infty) d\xi^i d\xi^j$.

According to the terms of the introduction of the Lagrange basis $\vartheta_1{}^i$ we have $ds^2 = ds_1{}^2$ by virtue of the continuum motion $x^i = x^i(\xi^j, t)$.

The reversible, irreversible and total components of the deformation are

$$\varepsilon^e_{ij} = {}^1/_2\,(g^{(1)}_{ij} - g^{(2)}_{ij}), \qquad \varepsilon^p_{ij} = {}^1/_2\,(g^{(2)}_{ij} - g^{(0)}_{ij}), \qquad \varepsilon_{ij} = {}^1/_2\,(g^{(1)}_{ij} - g^{(0)}_{ij}) \quad (1.1)$$

The space 2 is a space of final states for irreversible deformation, and a space of initial states for reversible deformation; the space 1 is a space of final states for reversible, as well as for the total components of the deformation. Let us introduce the tensors of reversible ε^e, irreversible ε^p, and total ε deformation

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\partial_1{}^i\partial_1{}^j, \quad \boldsymbol{\varepsilon}^e = \varepsilon_{ij}{}^e\partial_1{}^i\partial_1{}^j, \quad \boldsymbol{\varepsilon}^p = \varepsilon_{ij}{}^p\partial_2{}^i\partial_2{}^j$$

for the various deformation components in the spaces of final states.

Here $\varepsilon_{ij}(\xi^k, t)$, $\varepsilon_{ij}^e(\xi^k, t)$, $\varepsilon_{ij}^p(\xi^k, t)$ are defined by (1.1). On the basis of (1.1), for the components of these tensors defined in different spaces we will have the componentwise (matrix) equality

$$\varepsilon_{ij}{}^e + \varepsilon_{ij}{}^p = \varepsilon_{ij} \tag{1.2}$$

Let us apply the operation of "convective differentiation" in the time D/Dt for the constant Lagrange coordinates ξ^k to (1.2)

$$\frac{D\varepsilon_{ij}{}^e}{Dt} + \frac{D\varepsilon_{ij}{}^p}{Dt} = \frac{D\varepsilon_{ij}}{Dt} = e_{ij} \tag{1.3}$$

Let us define the strain rate tensors in the final states

$$\mathbf{e} = e_{ij}\partial_1{}^i\partial_1{}^j, \qquad \frac{D\boldsymbol{\varepsilon}^e}{Dt} = \frac{D\varepsilon_{ij}{}^e}{Dt}\partial_1{}^i\partial_1{}^j, \qquad \frac{D\boldsymbol{\varepsilon}^p}{Dt} = \frac{D\varepsilon_{ij}{}^p}{Dt}\partial_2{}^i\partial_2{}^j \tag{1.4}$$

Utilizing (1.4), we pass from the noninvariant (matrix) Eq. (1.3) to the tensor equation [2]. To do this we introduce the local basis $\partial_1{}^i$ in the unloading space 2. then denoting the components of the tensor $D\boldsymbol{\varepsilon}^p/Dt$ in the basis $\partial_1{}^i$ by $\gamma_{ij}{}^p$ we obtain

$$D\varepsilon_{ij}{}^p/Dt = C^{\alpha}_{\cdot i}\gamma^p_{\alpha\beta}C^{\beta}_{\cdot j}, \qquad \mathbf{C} = C^{\alpha}_{\cdot\beta}\partial_1{}^\beta\partial_{1\alpha} \tag{1.5}$$

Here the tensor C with matrix $\|C^{\alpha}_{\cdot\beta}\|$ defines the transformation from the covariant basis vector ∂_{2i} to the vector basis $\partial_{1\alpha}$ according to the law $\partial_{2i} = C^{\alpha}_{\cdot i}\partial_{1\alpha}$. The space 1 differs from the space 2 by elastic deformations $\mathbf{C}$ and elastic rotations of each particle of the medium, hence in the basis $\partial_1{}^i$ we have the following representation for the tensor $\mathbf{C}$ [2]:

$$\mathbf{C} = \exp[\mathbf{k}]\sqrt{\mathbf{g} - 2\boldsymbol{\varepsilon}^e}, \qquad \mathbf{k} = k_{ij}\partial_1{}^i\partial_1{}^j, \qquad \mathbf{g} = g_{ij}{}^{(1)}\partial_1{}^i\partial_1{}^j \tag{1.6}$$

Here $\mathbf{k}$ is the antisymmetric tensor of elastic rotations; $\mathbf{g}$ is the fundamental metric tensor. Substituting (1.6) into (1.3), we obtain the tensor equation (*)

$$D\boldsymbol{\varepsilon}^e/Dt + (\mathbf{g} - 2\boldsymbol{\varepsilon}^e)^{1/2}\exp[-\mathbf{k}]\boldsymbol{\gamma}^p\exp[\mathbf{k}](\mathbf{g} - 2\boldsymbol{\varepsilon}^e)^{1/2} = \mathbf{e} \tag{1.7}$$

Passing from the frozen ξ^i to the fixed x^k system, taking account of the transformations for convective derivatives [2 and 10], we have (1.8)

$$d\boldsymbol{\varepsilon}^e/dt + \boldsymbol{\omega}\boldsymbol{\varepsilon}^e - \boldsymbol{\varepsilon}^e\boldsymbol{\omega} + \mathbf{e}\boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^e\mathbf{e} + (\mathbf{g} - 2\boldsymbol{\varepsilon}^e)^{1/2}\exp[-\mathbf{k}]\boldsymbol{\gamma}^p\exp[\mathbf{k}](\mathbf{g} - 2\boldsymbol{\varepsilon}^e)^{1/2} = \mathbf{e}$$

Here the tensors $\boldsymbol{\varepsilon}^e$, $\mathbf{e}$, $\mathbf{g}$, $\mathbf{k}$, γ^p, ω are defined in the x^k system and have the covariant components $\varepsilon_{ij}{}^e$, e_{ij}, g_{ij}, k_{ij}, $\gamma_{ij}{}^p$, ω_{ij}, where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^\alpha\nabla_\alpha, \qquad e_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i), \qquad \omega_{ij} = \frac{1}{2}(\nabla_i v_j - \nabla_j v_i)$$

v^α are velocity vector components, e_{ij} strain rate tensor components, $\|\omega_{ij}\|$ the matrix of the vorticity tensor, ∇_α the symbol of convariant differentiation.

The kinematic relationship (1.8) defines the desired connection between the elastic, irreversible, and total tensor characteristics of the deformation. In contrast to the matrix relationship (1.3), the irreversible strain rate in the tensor relationship (1.7) is connected nonlinearly (because of the reversible deformations, and elastic rotations of an element of the medium) to the total strain rate and the rate of elastic strain.

Later we shall consider only such kinds of media whose macroscopic state is independent of internal rotations, and therefore of the quantity $\mathbf{k}$. As will be seen later, governing equations of such media, without the tensor $\mathbf{k}$, may actually be obtained.

Let us introduce the new tensor

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*) Iu.A. Buevich has obtained an analogous equation, where the kinematics of finite elastoplastic deformations is considered somewhat differently for Maxwellian media.

$$\mathbf{e}^p = \exp[-\mathbf{k}]\,\boldsymbol{\gamma}^p \exp[\mathbf{k}] = e_{ij}{}^p \ni^i \ni^j \tag{1.9}$$

It follows from (1.9) and the symmetry of γ^p that $\mathbf{e}^p$ is symmetric tensor; all three invariants of $\mathbf{e}^p$ coincide with the invariants of γ^p, however the principal directions differ by the magnitude of the elastic rotations.

It is convenient to take the Hencky tensor $\mathbf{h}$, which is an isotropic function of the tensor $\boldsymbol{\varepsilon}^e$, as a measure of the reversible deformation:

$$\mathbf{h} = -{}^1/_2 \ln(\mathbf{g} - 2\boldsymbol{\varepsilon}^e) \tag{1.10}$$

The principal axes of the tensors $\mathbf{h}$ and $\boldsymbol{\varepsilon}^e$ coincide.

Inserting the quantity $\mathbf{e}^p$ according to (1.9) into the fundamental kinematic relation (1.8), replacing $\boldsymbol{\varepsilon}^e$ by $\mathbf{h}$ according to (1.10), and multiplying on the left and on the right in this equation by the nondegenerate matrix $\exp[\mathbf{h}]$, we obtain

$$\frac{\Delta\mathbf{h}}{\Delta t} + \mathbf{e}^p - \mathbf{e} = \mathbf{f}\left(\mathbf{h};\ \boldsymbol{\omega},\ \mathbf{e},\ \frac{d\mathbf{h}}{dt}\right) \tag{1.11}$$

$$2\mathbf{f} = \exp[\mathbf{h}]\,\frac{d}{dt}(\exp[-2\mathbf{h}])\exp[\mathbf{h}] + 2\,\frac{d\mathbf{h}}{dt} + 2\boldsymbol{\omega}\mathbf{h} -$$
$$- 2\mathbf{h}\boldsymbol{\omega} + \exp[\mathbf{h}]\,\boldsymbol{\omega}\exp[-\mathbf{h}] - \exp[-\mathbf{h}]\,\boldsymbol{\omega}\exp[\mathbf{h}] +$$
$$+ \exp[\mathbf{h}]\,\mathbf{e}\exp[-\mathbf{h}] + \exp[-\mathbf{h}]\,\mathbf{e}\exp[\mathbf{h}] - 2\mathbf{e}$$

Here and henceforth, tensor (matrix) products are introduced. The Jaumann derivative

$$\left(\frac{\Delta h}{\Delta t}\right)_{ij} = \frac{\partial h_{ij}}{\partial t} + v^\alpha \nabla_\alpha h_{ij} + \omega_{i\cdot}^{\cdot\alpha} h_{\alpha j} - h_{i\alpha}\omega_{\cdot j}^{\alpha\cdot} \tag{1.12}$$

is denoted by the symbol $\Delta/\Delta t$.

The distinguishing property of the Jaumann derivative is

$$(\Delta\mathbf{g}/\Delta t)_{ij} = 0 \tag{1.13}$$

The tensor $\mathbf{f}$ from (1.11) possesses the following properties: $\mathbf{f}$ is a symmetric tensor, i.e., $f_{ij} = f_{ji}$.

The scalar product of the tensor $\mathbf{f}$ by an arbitrary function $\phi(\mathbf{h})$ is zero, i.e.,

$$\varphi^{ij}(h_{\alpha\beta})\,f_{ij} = 0, \qquad g^{ij} f_{ij} = \operatorname{Sp}\mathbf{f} = 0 \tag{1.14}$$

For sufficiently small elastic strains ($\mathbf{h} = \alpha\mathbf{H}$, $\alpha \ll 1$)

$$2\mathbf{f} = \mathbf{h}^2\mathbf{e} - 2\mathbf{heh} + \mathbf{eh}^2 + O(\mathbf{h}^3\mathbf{e} + \ldots) +$$
$$+ \mathbf{h}\boldsymbol{\omega}\mathbf{h}^2 - \mathbf{h}^2\boldsymbol{\omega}\mathbf{h} + {}^1/_3\,\mathbf{h}^3\boldsymbol{\omega} - {}^1/_3\,\boldsymbol{\omega}\mathbf{h}^3 + O(\mathbf{h}^4\boldsymbol{\omega} + \ldots) +$$
$$+ \frac{2\mathbf{h}}{3}\frac{d\mathbf{h}}{dt}\mathbf{h} - \frac{\mathbf{h}^2}{3}\frac{d\mathbf{h}}{dt} - \frac{d\mathbf{h}}{dt}\frac{\mathbf{h}^2}{3} + O\left(\mathbf{h}^3\frac{d\mathbf{h}}{dt} + \ldots\right) \tag{1.15}$$

Taking account of (1.11) and (1.12), Formula (1.15) shows that for sufficiently small reversible deformations the right side of the kinematic relation (1.11) contains terms two orders higher than the terms of the left side.

When the kinematic tensors ω, $\mathbf{e}$ and $d\mathbf{h}/dt$ commute with the tensor $\mathbf{h}$, $\mathbf{f} \equiv 0$ holds. Such a case is realized, say, in affine deformations of the medium, when the directions of the principal axes of the tensors $\mathbf{e}$, $\mathbf{h}$ and $d\mathbf{h}/dt$ coincide or are fixed in space, and $\boldsymbol{\omega} = 0$.

Contracting the kinematic relation (1.11) according to subscripts, we obtain

$$dh_{\alpha\alpha}/dt + \gamma_{\alpha\alpha}^p = e_{\alpha\alpha} \tag{1.16}$$

Introducing the notation ρ_0, ρ_1, ρ_2; g_0, g_1, g_2 for the densities and determinants of the metric tensors in the initial, deformed and "unloading" states, respectively, we will have

$$h_{\alpha\alpha} = \frac{1}{2}\ln\frac{g_1}{g_2} = \ln\frac{\rho_2}{\rho_1}, \quad \gamma_{\alpha\alpha}^p = \frac{1}{2}\frac{d}{dt}\ln\frac{g_2}{g_0} = \frac{d}{dt}\ln\frac{\rho_0}{\rho_2}$$

$$e_{\alpha\alpha} = \frac{1}{2}\frac{d}{dt}\ln\frac{g_1}{g_0} = \frac{d}{dt}\ln\frac{\rho_0}{\rho_1} \tag{1.17}$$

Substituting these expressions into (1.16), we obtain an identity of obvious physical meaning: the sum of the reversible and irreversible volume strain rates equals the total volume strain rate of the medium.

As will be shown below, the introduced tensor $e_{ij}{}^p$ is defined uniquely in terms of the observed kinematic (strain rate tensor e_{ij}) and dynamic (stress tensor σ_{ij}) quantities.

Hence, despite the fact that the tensor of elastic rotations k_{ij} remains undefined in terms of these quantities, components of the irreversible strain rate tensor in the unloading space

$$D\varepsilon^p/Dt = \exp\ [-\mathbf{h}]\ \mathbf{e}^p \exp\ [-\mathbf{h}]$$

can easily be determined by means of the transformation formulas (1.5), (1.6), taking account of the definition of $e_{ij}{}^p$.

Let us note that the kinematic relations (1.8) and (1.11) in the two limit cases $\varepsilon^e \to 0$ $(\mathbf{k} \to 0)$ or $\gamma^p \to 0$ go over into the kinematic relations for a viscous fluid and a medium with reversible elastic strains, respectively.

2. Expression for entropy production in a system. Simplest viscoelastic models. Common to any type of continuum are the equations of conservation of mass, momentum and total energy

$$\frac{d\rho}{dt} = -\rho\frac{\partial v_\beta}{\partial x_\beta}, \qquad \rho\frac{dv_i}{dt} = \frac{\partial\sigma_{i\beta}}{\partial x_\beta}, \qquad \rho\frac{dw}{dt} = \frac{\partial}{\partial x_\beta}\,(v_\alpha\sigma_{\alpha\beta} - q_\beta) \tag{2.1}$$

Here ρ is the density of the medium, v_α the velocity vector components, σ_{ij} the stress tensor components, w the total energy of unit mass, q_β the heat flux vector components. For simplicity the equations are written in a Cartesian rectangular coordinate system.

The stress tensor in a medium without internal moments is symmetric $\sigma_{ij} = \sigma_{ji}$, and the total energy consists of the kinetic energy and the internal energy of the medium $\rho w = \frac{1}{2}\rho v_\alpha^2 + \rho u$.

The principal difference in the various model media is in the specific internal energy u. As has already been said, we shall consider that medium in which the specific internal energy depends on the specific entropy s and the Hencky tensor h of the reversible deformation $u = u(s, h_{ij})$ (here the choice of h_{ij} instead of $\varepsilon_{ij}{}^e$ is made from considerations of convenience). The Gibbs relation may be written as

$$\frac{du}{dt} = T\frac{ds}{dt} + \frac{du}{dt}\bigg|_s \tag{2.2}$$

Utilizing the equations of this Section it is easy to obtain an equation for the specific entropy

$$\rho\frac{ds}{dt} = -\frac{\partial}{\partial x_\beta}\frac{q_\beta}{T} + P_s, \quad P_s \geqslant 0, \quad TP_s = -\frac{q_\beta}{T}\frac{\partial T}{\partial x_\beta} + \sigma_{\alpha\beta}e_{\alpha\beta} - \rho\frac{du}{dt}\bigg|_s \tag{2.3}$$

Here $e_{\alpha\beta}$ is the strain rate, P_s the entropy production, which according to the second law of thermodynamics is positive for nonequilibrium processes and vanishes at equilibrium. The uniqueness of the isolation of the expression P_s as the entropy production in (2.3) is based on the invariance of this expression relative to the Galileo transformation, and on P_s vanishing for thermodynamic equilibrium [12]. In the case of an isotropic medium, the scalar function of the internal energy may depend only on invariants of the strain tensor

$$I_1 = h_{\alpha\alpha}, \qquad I_2 = h_{\alpha\beta}h_{\beta\alpha}, \qquad I_3 = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}$$

Then $du/dt|_s$ can be written as

$$\rho\frac{du}{dt}\bigg|_s = \sigma_{\alpha\beta}^e\left(\frac{\Delta\mathbf{h}}{\Delta t}\right)_{\alpha\beta}, \qquad \frac{\sigma_{ij}^e}{\rho} = \frac{\partial u}{\partial I_1}\delta_{ij} + \frac{\partial u}{\partial I_2}2h_{ij} + \frac{\partial u}{\partial I_3}3h_{i\alpha}h_{\alpha j} \tag{2.4}$$

Here the σ_{ij}^e are components of the "elastic stress" tensor.

In order to extract the independent thermodynamic forces and thermodynamic fluxes correctly in the expression for the entropy production let us utilize the fundamental kinematic relationship (1.11), and let us divide the tensor quantities into global and deviatoric parts. For example

$$\sigma_{ij} = \sigma_{ij}' + {}^1/_3\sigma_{\alpha\alpha}\delta_{ij}, \qquad \sigma_{\alpha\alpha}' = 0$$

Then the expression for the entropy production is rewritten as

$$TP_s = (\sigma_{\alpha\beta}' - \sigma_{\alpha\beta}^{e'}) e_{\alpha\beta}^{e'} + \sigma_{\alpha\beta}^{e'} e_{\alpha\beta}^{p'} - \frac{q_\beta}{T}\frac{\partial T}{\partial x_\beta} + {}^1/_3 (\sigma_{\alpha\alpha} - \sigma_{\alpha\alpha}^e) e_{\beta\beta} + {}^1/_3\sigma_{\alpha\alpha}^e e_{\beta\beta}^p \quad (2.5)$$

It is now possible to proceed to obtaining the governing equations of the medium. For a thermodynamic approach to describing it, the quantities $\sigma_{\alpha\beta}$, $\sigma_{\alpha\beta}^e$ and $\partial T/\partial x_\beta$ in (2.5) for the entropy production play the part of thermodynamic forces, $e_{\alpha\beta}$, $e_{\alpha\beta}^p$ and q_β, of thermodynamic fluxes. According to the customary linear theory of TIP they are connected by linear phenomenological relationships [12], which, in particular, yield the governing equations of the medium.

By virtue of the Curie principle, the phenomenological relationships for scalar, vector, and tensor phenomena separate in an isotropic medium. Taking account of the Onsager reciprocity relation [12], we obtain for the scalar phenomena

$$\sigma_{\alpha\alpha} - \sigma_{\alpha\alpha}^e = a_1 e_{\alpha\alpha} + a_2 e_{\alpha\alpha}^p, \qquad \sigma_{\alpha\alpha}^e = a_2 e_{\alpha\alpha} + a_3 e_{\alpha\alpha}^p \quad (2.6)$$

for the vector phenomena

$$q_i = -\varkappa\,(\partial T / \partial x_i) \quad (2.7)$$

for the tensor phenomena

$$\sigma_{ij}' - \sigma_{ij}^{e'} = b_1 e_{ij}' + b_2 e_{ij}^{p'}, \qquad \sigma_{ij}^{e'} = b_2 e_{ij}' + b_3 e_{ij}^{p'} \quad (2.8)$$

The kinetic coefficients $\varkappa$, a_k, b_k are generally functions of T and $I_k(h_{ij})$.

Entropy production becomes a nonnegative-definite quadratic form (α_k, β_k are easily expressed in terms of a_k, b_k) (2.9)

$$TP_s = \alpha_1 e_{\alpha\alpha}^2 + 2\alpha_2 e_{\alpha\alpha}\sigma_{\beta\beta} + \alpha_3\sigma_{\beta\beta}^2 + \beta_1 e_{\alpha\beta}'^2 + 2\beta_2 e_{\alpha\beta}'\sigma_{\alpha\beta}' + \beta_3\sigma_{\alpha\beta}'^2 + \varkappa T^{-1}(\partial T/\partial x_\beta)^2$$

Conditions for positive-definiteness of the quadratic form are

$$\varkappa > 0,\ a_1 > 0,\ b_1 > 0,\ a_1 a_3 > a_2^2,\ b_1 b_3 > b_2^2,\ \alpha_1 > 0,\ \beta_1 > 0$$

$$\alpha_1\alpha_3 > \alpha_2^2, \quad \beta_1\beta_3 > \beta_2^2 \quad (2.10)$$

The inequalities (2.10) (part of which may be weakened in various particular cases) are sufficient also for a unique definition of the flows in terms of the thermodynamic forces.

Taking account of the inequalities (2.10), the kinematic relationship (1.11), and the expressions (2.4) for σ_{ij}^e, Eqs. (2.6) and (2.8) are a closed nonlinear system of rheological equations of some isothermal model of a compressible viscoelastic fluid, which is as shown below, describes retardation and relaxation.

Let us consider the equation for the medium temperature.

Let us determine the specific heat for a constant reversible deformation

$$c_h = \left(\frac{\partial u}{\partial T}\right)_h = T\left(\frac{\partial s}{\partial T}\right)_h \quad (2.11)$$

Transforming it by utilizing (2.3), we obtain

$$\rho c_h \frac{dT}{dt} + \rho T\left(\frac{\partial s}{\partial h_{\alpha\beta}}\right)_T \frac{dh_{\alpha\beta}}{dt} = \nabla_\alpha(\varkappa\nabla_\alpha T) + TP_s^+ \quad (2.12)$$

$$TP_s^+ = TP_s - \varkappa(\nabla_\beta T)^2$$

From the condition of integrability of the specific free energy $u - Ts$ we have

$$\left(\frac{\partial s}{\partial h_{ij}}\right)_T = \left(\frac{\partial \sigma_{ij}^e/\rho}{\partial T}\right)_h$$

Then (2.12) may be transformed into

$$\rho c_h \frac{dT}{dt} = \nabla_\alpha (\varkappa \nabla_\alpha T) + TP_s^+ + \rho T \left(\frac{\partial \sigma_{\alpha\beta}^e/\rho}{\partial T}\right)_h \frac{dh_{\alpha\beta}}{dt} \qquad (2.13)$$

Formula (2.13) shows that the thermal effect in the deformation of the viscoelastic medium considered is due to the dissipative term, as well as an additional term, which appear particularly sharply in rapid changes of the mode of medium deformation. Great heating has actually been observed [14] in rotational viscosimeters during a sudden stop in the flow of viscoelastic fluids of various kinds.

We can introduce the specific heat for a constant tensor $\tau = \sigma^e/\rho$ (which corresponds to constant stress in Maxwellian or elastic media), which is connected with c_h by means of the relationship

$$c_\tau = c_h - T\left(\frac{\partial \tau_{\alpha\beta}}{\partial T}\right)_\tau \frac{d\tau_{\alpha\beta}}{dt}$$

The heat conduction Eq. (2.13) becomes

$$\rho c_\tau \frac{dT}{dt} = \nabla_\alpha (\varkappa \nabla_\alpha T) + TP_s^+ + \rho T \left(\frac{\partial h_{\alpha\beta}}{\partial T}\right)_\tau \frac{d\tau_{\alpha\beta}}{dt} \qquad (2.14)$$

When the elasticity in the medium is of entropic nature, as may be in the flow of polymer solutions and melts, for isothermal deformation

$$0 = \left(\frac{\partial u}{\partial h_{ij}}\right)_T = \left(\frac{\partial u}{\partial s}\right)_h \left(\frac{\partial s}{\partial h_{ij}}\right)_T + \left(\frac{\partial u}{\partial h_{ij}}\right)_s = -T\left(\frac{\partial \tau_{ij}}{\partial T}\right)_h + \tau_{ij}$$

From this results $\tau_{ij} = \tau_{ij}^\circ T/T_0$ (the superscript $^\circ$ shows that the tensor τ_{ij}° is referred to some "initial" temperature T_0). In combination with the above-mentioned rheological Eqs. (2.13) or (2.14) describe nonisothermal behavior of the considered viscoelastic medium.

The system of Eqs. (1.11), (2.4), (2.6) and (2.8) describes the nonlinear behavior of a medium possessing stress relaxation and aftereffect. Let us show that in particular cases the nonlinear Maxwell model with relaxation time, and the Kelvin-Voigt model with retardation time can be obtained from these equations.

1°. Nonlinear Maxwell Model. Let us set

$$a_1 = a_2 = b_1 = b_2 = 0, \qquad a_3 > 0, \qquad b_3 > 0$$

in (2.6) and (2.8).

Then using the notation $b_3 = 2\eta$, $a_3 = 3\zeta$ (η, ζ shear and volume viscosity coefficients), we obtain

$$\sigma_{ij} = 2\eta e_{ij}^p + (\zeta - {}^2/_3\,\eta)\, e_{\alpha\alpha}^p \delta_{ij} = \sigma_{ij}^e = \rho \frac{\partial u}{\partial I_1}\delta_{ij} + 2\rho\frac{\partial u}{\partial I_2} h_{ij} + 3\rho \frac{\partial u}{\partial I_3} h_{i\alpha} h_{\alpha j} \quad (2.15)$$

The system (2.15) shows that the stress tensor in a Maxwell fluid is connected with the elastic strain tensor just as in the equilibrium case of a purely elastic medium, and the tensor e_{ij}^p characterizing the irreversible strain rate is also connected with the stresses by Newton's law, as in the case of a viscous medium.

The expression for the dissipation takes the simple form

$$TP_s - \varkappa T^{-1} (\nabla_\alpha T)^2 = \zeta e_{\alpha\alpha}^{p2} + 2\eta e_{\alpha\beta}^{p\prime 2} = \sigma_{\alpha\alpha}^2 / (9\zeta) + (\sigma_{\alpha\beta}')^2 / (2\eta) \qquad (2.16)$$

As follows from Section 1, $e_{\alpha\beta}^{p\prime 2} = \gamma_{\alpha\beta}^{p\prime 2}$ holds, i.e., the elastic rotations of elements of the medium do not affect the value of the dissipation. If $\mathbf{e}^p$, $\mathbf{h}$ are expressed in terms of σ according to (2.15) and substituted into the kinematic relationship (1.11), we then obtain the rheological equation of a Maxwell fluid, which connects the stress tensor with the total strain rate tensor.

2°. Nonlinear Kelvin-Voigt Model (see also [15]). This model can be obtained by the following formal means. Let us set $e^p = 0$ ($\gamma^p = 0$) in (2.5) and the kinematic relationship (1.11). Then we have $\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha}^{e} + ae_{\alpha\alpha}$ in place of (2.6). Analogously, $\sigma_{ij}' = \sigma_{ij}'^{e} + be_{ij}'$. Furthermore, let us use the notation $a = 3\zeta$, $b = 2\eta$. For $\gamma^p = 0$ there holds $\varepsilon^e = \varepsilon$ and the kinematic relationship (1.11) becomes (see also (1.8))

$$\frac{d\varepsilon}{dt} + \omega\varepsilon - \varepsilon\omega + e\varepsilon + \varepsilon e = e \tag{2.17}$$

This is the expression for the customary connection between the finite strain tensor and the strain rate. The corresponding rheological equation will be

$$\sigma = \sigma^e(h) + 2\eta e + (\zeta - {}^2/_3\,\eta)(Sp\, e)\, g \tag{2.18}$$

In combination with the heat conduction equation, the system of Eqs. (2.17) and (2.18) is a closed system of thermorheological equations for a compressible viscoelastic isotropic medium with aftereffect. The expression for the entropy production is

$$TP_s = \varkappa T^{-1}(\nabla_\alpha T)^2 + \zeta e_{\alpha\alpha}^2 + 2\eta e_{\alpha\beta}'^2 \tag{2.19}$$

A relationship of the type (2.18) has been obtained in [15] for the case of large elastic deformations.

In concluding this Section let us make two remarks.

1. Phenomenological connections between the stresses, total strains and their total time derivatives, obtained on the basis of an expression of type (2.4) for the entropy production without the kinematic relationship (1.11), become very ambiguous. This latter follows, say, from the fact that

$$\varphi_{ij}(\varepsilon)\left(\frac{\Delta\psi(\varepsilon)}{\Delta t}\right)_{ij} = \varphi_{ij}(\varepsilon)\,\psi_{jk}'(\varepsilon)\left(\frac{\Delta\varepsilon}{\Delta t}\right)_{ki} = \varphi_{ij}(\varepsilon)\,\psi_{jk}'(\varepsilon)\,\frac{d\varepsilon_{ki}}{dt}$$

while

$$\left(\frac{\Delta\psi(\varepsilon)}{\Delta t}\right)_{ij} \neq \psi_{ik}'(\varepsilon)\left(\frac{d\varepsilon}{dt}\right)_{kj}$$

The arbitrariness in selecting the thermodynamic forces which appears in the absence of the kinematic relationship leads to great arbitrariness in the rheological relations obtained.

In the presence of the kinematic relationships (1.11), independently of the selection of the measure of reversible strain, the final rheological equations are obtained completely uniquely as a result of the above-mentioned procedure.

2. In general, the results of this Section refer to the case of weak nonequilibrium; it can only be hoped (as the examples presented below indicate) that they have a sufficiently broad domain of applicability for viscoelastic media. In the more general case it is apparently expedient to use the methods elucidated in [11].

3. Governing equations for simple viscoelastic fluids in the presence of sufficiently small reversible strains. Let the reversible strains in a viscoelastic medium be sufficiently small as compared with the total strains. Such a case is realized in weakly elastic fluids as well as for sufficiently slow motions. Formally expanding the kinematic relationship (1.11) for sufficiently small $\mathbf{h}$ and discarding terms whose order is $\mathbf{h}^2$ greater than the rest, we will have the "linearized" kinematic relationship

$$\Delta h / \Delta t + e^p = e \tag{3.1}$$

For sufficiently small $\mathbf{h}$ the function $u(s, \mathbf{h})$ can be represented with cubic accuracy as

$$\rho_0 u = \rho_0 u_0(s) + \mu I_2 + {}^1/_2\lambda_0 I_1^2 + {}^1/_3\,\lambda_1 I_3 + \lambda_2 I_1 I_2 + {}^1/_3\lambda_3 I_1^3 \tag{3.2}$$

Here ρ_0 is the value of the medium density in the undeformed state at the temperature T_0; μ is the shear modulus; $K = \lambda_0 + 2/3\mu$ the modulus of multilateral compression; $\lambda_1, \lambda_2, \lambda_3$ the characteristics of the "anharmonic part" of the internal energy.

According to the requirement for thermodynamic stability of the system, the expansion of

u in terms of $\mathbf{h}$ starts with quadratic terms in which μ and K are positive, but the signs of the λ_n are not definite.

Now evaluating σ_{ij}^e on the basis of (2.4) and utilizing (3.2), we have

$$\rho_0 / \rho \sigma_{ij}^e = (\lambda_0 I_1 + \lambda_2 I_2 + \lambda_3 I_1^2)\,\delta_{ij} + 2\,(\mu + \lambda_2 I_1)\,h_{ij} + \lambda_1 h_{i\alpha} h_{\alpha j} \qquad (3.3)$$

It is seen from this expression that keeping just third order anharmonic terms in the expansion of the internal energy in terms of the strain $\mathbf{h}$ corresponds to the accuracy of the "linearized" kinematic relationship (3.1). As in (3.1), there are lower order terms in $\mathbf{h}$ in (3.3) and terms whose order is $\mathbf{h}^2$ greater than the rest are not taken into account (*)

It is easy to separate σ^e into spherical and deviatoric parts

$$\rho_0 / \rho \sigma_{\alpha\alpha}^e = (3\lambda_0 + 2\mu)\,I_1 + (3\lambda_2 + \lambda_1)\,I_2 + (2\lambda_2 + 3\lambda_3)\,I_1^2$$

$$\rho_0 / \rho \sigma_{ij}^{e\prime} = 2\,(\mu + \lambda_2 I_1)\,h_{ij}' + \lambda_1 (h_{i\alpha} h_{\alpha j} - {}^1/_3 I_2 \delta_{ij}) \qquad (3.4)$$

Relative to the spherical part it is reasonable to expect that at low pressures the irreversible volume changes are insignificant, i.e., $e_{\alpha\alpha}^p \approx 0$. Then according to (1.16) and (2.6)

$$\sigma_{\alpha\alpha} - \sigma_{\alpha\alpha}^e = a e_{\alpha\alpha} = dh_{\alpha\alpha} / dt$$

Utilizing (3.4), $\sigma_{\alpha\alpha}^e$ can hence be eliminated, and an equation relating $\sigma_{\alpha\alpha}$ and $I_1 = h_{\alpha\alpha}$ can be obtained:

$$\sigma_{\alpha\alpha} = a\,(dI_1 / dt) + \frac{\rho_0}{\rho}\,[(3\lambda_0 + 2\mu)\,I_1 + 3(\lambda_2 + \lambda_1)\,I_2 + (2\lambda_2 + 3\lambda_3)\,I_1^2] \qquad (3.5)$$

This equation describes the volume aftereffect in the medium. Since $e_{\alpha\alpha}^p = 0$ we have $I_1 = h_{\alpha\alpha} = \ln(\rho_0/\rho)$, then for small deformations (3.5) passes into the nonlinear Kelvin-Voigt equation relating the volume strain to the isotropic pressure. In the more general case, when it is impossible to neglect irreversible volume changes ($e_{\alpha\alpha}^p \neq 0$), it is necessary to use the system of Eqs. (1.16), (2.1), (2.6) and (3.4), which describes relaxation of the pressure and strain rate, in order to describe volume effects.

Let us here consider the simplest case also for the deviatoric stresses. Let us assume that in the expression for internal energy the anharmonic terms may generally be neglected (**). Then Hooke's law $\sigma_{ij}^{e\prime} = 2\mu h_{ij}'$ holds for the "elastic" stresses, and it is easy to eliminate $\mathbf{e}^p$ and $\mathbf{h}$ from the system of Eqs. (2.8). We hence obtain the rheological Eq.

$$\theta_1 \left(\frac{d\sigma_{ij}'}{dt} - \sigma_{i\alpha}'\omega_{\alpha j} + \omega_{i\alpha}\sigma_{\alpha j}'\right) + \sigma_{ij}' = 2\eta \left[\theta_2 \left(\frac{de_{ij}'}{dt} + \omega_{i\alpha} e_{\alpha j}' - e_{i\alpha}'\omega_{\alpha j}\right) + e_{ij}'\right] \qquad (3.6)$$

In deriving (3.6) the case of an incompressible medium ($\rho = \rho_0$) was considered and the coefficients in relationships (2.8) were assumed constant. The coefficients θ_1, θ_2 and η in (3.6) are connected with the coefficients of (2.8) as follows:

$$\theta_1 = b_3 / (2\mu), \qquad \theta_2 = (b_1 b_3 - b_2^2) / (4\mu\eta), \qquad 2\eta = b_1 + 2b_2 + b_3 \qquad (3.7)$$

It follows from the inequalities (2.10) and from $\mu > 0$ that

$$\eta > 0, \qquad \theta_1 > \theta_2 > 0 \qquad (3.8)$$

i.e., the positiveness of the viscosity coefficient and the relaxation time, and the fact that the stress relaxation time is always greater than the time of the after-effect of the strain rate.

An illustrative model of one elastic spring and two viscous elements in two equivalent

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*) It is understood that only terms which do not raise the order of (3.3) as a whole are kept in the expansion of ρ in terms of $\mathbf{h}$.

**) Within the scope of the expounded phenomenological theory, the satisfactoriness of such an approximation was not clear before; comparison of the consequences of the model equations with experimental results will be elucidated below.

versions corresponds to the linearized Eq. (3.6) at low flow rates: in one version the Maxwell element connected in parallel with one of the viscous elements, can be isolated, and in the other the Kelvin-Voigt element, connected in series with a viscous element.

Model equations similar to (3.6) have been repeatedly relied upon, and not without success, for the description of viscoelastic fluids [16]. However, as will be shown below, these equations do not describe correctly enough such an important property of viscoelastic fluids as the normal stresses. In those cases when the effect of the normal stresses plays a large part, refined equations are necessary.

There are several formal possibilities for refining the equations by remaining within the scope of the kinematic relationship (3.1) linear in **h**. Two of them concern refinement of the phenomenological relationships between thermodynamic fluxes. One is to keep quadratic terms in the thermodynamic forces (or fluxes) in relations of the type (2.6) to (2.8); such a method extends beyond the scope of customary TIP theory.

The other method within the limits of this theory is to take account of the dependence of the kinetic coefficients on the elastic strains **h**. Elastic strains result in a deformational anisotropy of an initially isotropic medium so that it is necessary to generalize the relationships (2.6) to (2.8). In the incompressible case ($e_{\alpha\alpha}=0$, $h_{\alpha\alpha}=0$), the relationships for the stresses become, with linear accuracy in the strain **h**,

$$\sigma_{ij}-\sigma_{ij}^{e}=b_1e_{ij}+b_{11}(h_{i\alpha}e_{\alpha j}+h_{j\alpha}e_{\alpha i})+b_{12}h_{\alpha\beta}e_{\alpha\beta}\delta_{ij}+$$
$$+b_2e_{ij}^{p}+b_{21}(h_{i\alpha}e_{\alpha j}^{p}+e_{i\alpha}^{p}h_{\alpha j})+b_{22}h_{\alpha\beta}e_{\alpha\beta}^{p}\delta_{ij} \qquad (3.9)$$

$$\sigma_{ij}^{e}=b_2e_{ij}+b_{21}(h_{i\alpha}e_{\alpha j}+e_{i\alpha}h_{\alpha j})+b_{22}h_{\alpha\beta}e_{\alpha\beta}\delta_{ij}+$$
$$+b_3e_{ij}^{p}+b_{31}(h_{i\alpha}e_{\alpha j}^{p}+e_{i\alpha}^{p}h_{\alpha j})+b_{32}h_{\alpha\beta}e_{\alpha\beta}^{p}\delta_{ij}$$

Still another possibility is to take account of the anharmonic members in the expression for the internal energy. As an illustration, let us write down the equation for an incompressible medium by utilizing the relationship (2.8) with constant kinetic coefficients and elastic stresses (3.3)

$$\left(\frac{\Delta h}{\Delta t}\right)_{ij}+\frac{1}{\theta_1}h_{ij}+\frac{\lambda_1}{b_3}\left(h_{i\alpha}h_{\alpha j}-\frac{1}{3}I_2\delta_{ij}\right)=\left(1+\frac{b_2}{b_3}\right)e_{ij}$$
$$\sigma_{ij}=(b_1-b_2^2/b_3)e_{ij}+\frac{b_2+b_3}{\theta_1}h_{ij}+\frac{\lambda_1}{b_3}\left[(b_2+b_3)h_{i\alpha}h_{\alpha j}-\frac{b_2}{3}I_2\delta_{ij}\right] \qquad (3.10)$$

The relationships (3.1), (3.4) and (3.9) possess the same accuracy in **h**. Within the limits of this same accuracy, they can be utilized to obtain an equation connecting **h** with the strain rate **e** for an incompressible medium

$$\left(\frac{\Delta h}{\Delta t}\right)_{ij}+\frac{2\mu}{b_3}h_{ij}+\frac{\lambda_1b_3-4\mu b_{31}}{2b_3^2}\{\mathbf{hh}\}_{ij}+\frac{b_2b_{31}-b_3b_{21}}{b_3^2}\{\mathbf{he}\}_{ij}=\frac{b_2+b_3}{b_3}e_{ij} \qquad (3.11)$$

The abbreviation $\{\mathbf{pq}\}_{ij}=p_{i\alpha}q_{\alpha j}+q_{i\alpha}p_{\alpha j}-{}^2/_3p_{\alpha\beta}q_{\alpha\beta}\delta_{ij}$ is used here. In deriving (3.11) it turns out to be necessary to impose the following constraints on the coefficients which originate from the condition of disappearance of the spherical parts in (3.11)

$$2b_{21}+3b_{22}=\frac{\lambda_1}{2\mu}b_2, \qquad 2b_{31}+3b_{32}=\frac{\lambda_1}{2\mu}b_3 \qquad (3.12)$$

Utilizing now (3.11) and (3.12) and the same relationships (3.11), (3.4) and (3.9), the expression for the stress σ_{ij} (after subtracting the isotropic pressure) can be written as

$$\sigma_{ij}=\left(b_1-\frac{b_2^2}{b_3}\right)e_{ij}+2\mu\left(1+\frac{b_2}{b_3}\right)h_{ij}+ \qquad (3.13)$$
$$+\frac{\lambda_1b_3(b_2+b_3)-4\mu(b_2b_{31}-b_3b_{21})}{2b_3^2}\{\mathbf{hh}\}_{ij}+\frac{\lambda_1}{3}\left(1+\frac{b_2}{b_3}\right)h_{\alpha\beta}^2\delta_{ij}+$$
$$+\left(b_{11}-2b_{21}\frac{b_2}{b_3}+b_{31}\frac{b_2^2}{b_3^2}\right)\{\mathbf{he}\}_{ij}+\left(b_{12}+\frac{2}{3}b_{11}-\frac{\lambda_1}{6\mu}\frac{b_2^2}{b_3}\right)h_{\alpha\beta}e_{\alpha\beta}\delta_{ij}$$

The obtained system of (3.11), (3.13) is the governing equations of the medium which connects $\boldsymbol{\sigma}$ and $\mathbf{e}$ by means of the tensor parameter $\mathbf{h}$. According to (2.5) the dissipation in such a model, to the accuracy of terms of two orders in $\mathbf{h}$, will become (isothermal case)

$$TP_s = \left(b_1 - \frac{b_2^2}{b_3}\right) e_{\alpha\beta}^2 + \frac{4\mu^2}{b_3} h_{\alpha\beta}^2 + \tag{3.14}$$

$$+ 4\mu \frac{\lambda_1 b_3 - 2\mu b_{31}}{b_3^2} h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} + 2\left(b_{11} - 2b_{21}\frac{b_2}{b_3} + b_{31}\frac{b_2^2}{b_3^2}\right) h_{\alpha\beta} e_{\beta\gamma} e_{\gamma\alpha}$$

Terms with coefficients b_{12}, b_{22}, b_{32} do not enter the expression for the dissipation in the considered incompressible case.

To the same quadratic accuracy as before, the parameter $\mathbf{h}$ can be eliminated from (3.11) and (3.13) and the equation connecting $\boldsymbol{\sigma}$ and $\mathbf{e}$ can be written explicitly

$$\theta_1 \left(\frac{\Delta s}{\Delta t}\right)_{ij} + s_{ij} + c_1\{\mathrm{ss}\}_{ij} + c_2\{\mathrm{se}\}_{ij} + c_3 s_{\alpha\beta}^2 \delta_{ij} + c_4 s_{\alpha\beta} e_{\alpha\beta} \delta_{ij} +$$

$$+ c_5\{\mathrm{ee}\}_{ij} + c_6 e_{\alpha\beta}^2 \delta_{ij} + c_7\left\{s\frac{\Delta \mathrm{e}}{\Delta t}\right\}_{ij} + c_8 s_{\alpha\beta}\left(\frac{\Delta \mathrm{e}}{\Delta t}\right)_{\alpha\beta}\delta_{ij} = 2\eta\left(1 - \frac{\theta_2}{\theta_1}\right) e_{ij} \tag{3.15}$$

$$s_{ij} = \sigma_{ij} - 2\eta\frac{\theta_2}{\theta_1} e_{ij}$$

The coefficients θ_1, θ_2, η, c_k are expressed in terms of the nine initial coefficients μ, λ_1, b_1, b_2, b_3, b_{11}, b_{12}, b_{21}, b_{31}. In the particular case when the coefficients of the last two members in (3.13) vanish, then $c_5 = c_6 = c_7 = c_8 = 0$ and the seven coefficients in (3.15) are expressed in terms of the seven original coefficients.

It is interesting to compare (3.15) with the equations of the Oldroyd eight-constant model [17]. Eqs. (3.15) possess a number of evident differences from the Eqs. in [17], for example, there are nonlinear terms in the stresses and terms with products of the stresses by the acceleration in (3.15). Moreover, despite the high arbitrariness in selecting the constants, (3.15) do not actually contain many particular cases admitted by the Oldroyd equations. Thus, by virtue of the existing relations between the coefficients c_k and b_{kl} it is impossible to obtain the equations of the "covariant" and "contravariant" models introduced by Oldroyd [10] from (3.15).

The exposition is here conducted under the assumption of isothermy. If there is a non-uniform temperature distribution T in the medium, this then results in the occurrence of a thermal flux q_i;

$$q_i = -(\varkappa_0 \delta_{i\alpha} + \varkappa_1 h_{i\alpha}) \nabla_\alpha T \tag{3.16}$$

The dependence of the heat conduction coefficient $\varkappa$ on the reversible strain $\mathbf{h}$ is taken into account in (3.16) in a linear approximation. The influence of the strain results in a deformational anisotropy of the heat conduction process even in a medium with isotropic structure. It is interesting that such a situation should concern not only solids but also elastic fluids. Analogously also for the diffusion process.

Let us now consider some particular cases admitted by the model Eqs. (3.11) and (3.13).

In the case of a linear connection between the "elastic stresses" and the reversible strains ($\lambda_1 = 0$), and if in addition we set $b_{12} = -\frac{1}{3} b_{11}$, equations of the same kind as in [18] are obtained. According to [18], the flow of weakly concentrated suspensions of slightly deformable elastic particles in a viscous fluid is described by such equations(*). It is easy to obtain equations in the same form as in [18], directly from (3.1), (3.4) and (3.9)

$$\left(\frac{\Delta \mathbf{h}}{\Delta t}\right)_{ij} + \frac{2\mu}{b_3} h_{ij} = \left(1 + \frac{b_2}{b_3}\right) e_{ij} + \frac{b_{21} + b_{31}}{b_3} (\mathbf{he})_{ij} - b_{31}\left\{\mathbf{h}\frac{\Delta \mathbf{h}}{\Delta t}\right\}_{ij} \tag{3.17}$$

*) Let us note that the specific numerical values for the coefficients in the equations from [18] do not agree with the possible values of the coefficients in (3.17).

$$\sigma_{ij} = (b_1 + 2b_2 + b_3) e_{ij} - (b_2 + b_3) \left(\frac{\Delta \mathbf{h}}{\Delta t}\right)_{ij} + $$
$$+ (b_{11} + 2b_{21} + b_{31}) \{\mathbf{he}\}_{ij} - (b_{21} + b_{31}) \left\{\mathbf{h} \frac{\Delta \mathbf{h}}{\Delta t}\right\}_{ij}$$

Since these equations have been obtained approximately both here and in [18], there is no special reason for keeping them in the form (3.17), and the first equation can be solved approximately for $\Delta \mathbf{h}/\Delta t$ and it is possible to go over to Eqs. of the form (3.11) and (3.13). In this case $c_3 = c_4 = c_6 = c_8 = 0$ in Eqs. of the form (3.15).

If we add $b_{31}(b_2 + b_3) = b_3 b_{21} - b_2 b_{31} = (b_3 b_{11} - b_2 b_{21})\, b_3/b_2 = \pm\ \tfrac{1}{3}\ (1 + \varepsilon)\, b_3^{\,2}$ to the previous assumptions about the coefficients ($\lambda_1 = 0$, $b_{12} = -\ 2/3\ b_{11}$), we thereby arrive at the model equations discussed in detail by Bird et al. (for example, [19] and [20])

$$\theta_1 \left(\frac{\Delta \mathbf{s}}{\Delta t}\right)_{ij} \mp \theta_1 (1 + \varepsilon) \{\mathbf{se}\}_{ij} + s_{ij} = 2\eta \left(1 - \frac{\theta_2}{\theta_1}\right) e_{ij}, \quad s_{ij} = \sigma_{ij} - 2\eta \frac{\theta_2}{\theta_1} e_{ij} \qquad (3.18)$$

The peculiarity of this model is that the terms $\{\mathbf{se}\}$ vanish in the case of a Maxwell type model ($\sigma_{ij} = \sigma_{ij}^{\,*}$) while the formal equality $\theta_2 = 0$ does not affect these terms (for $\theta_2 = 0$, Eq. (3.18) describes only stress relaxation, which is customarily associated with Maxwell type models). The reason is that $b_{21} \neq 0$ for $b_1 = b_2 = 0$ ($\theta_2 = 0$) by virtue of the constraints imposed earlier on the coefficients, and therefore, $\sigma_{ij} \neq \sigma_{ij}^{\,*}$ because of the nonlinear terms. Another peculiarity is that in such a model $\sigma_{\alpha\alpha} = 0$ in one-dimensional steady-state shear flows. It is possible to branch off easily from this by considering $b_{12} \neq -\ \tfrac{2}{3}\ b_{11}$, then spherical terms of the form

$$e_{\alpha\beta}^{\,2}\delta_{ij}, \quad e_{\alpha\beta} s_{\alpha\beta}\delta_{ij}, \quad s_{\alpha\beta} \quad (\Delta \mathbf{e}/\Delta t)_{\alpha\beta}\, \delta_{ij}$$

will appear in the equations.

For the model with Eqs. (3.18) with $T = \text{const}$, the expression for entropy production to the accuracy of terms of two orders is

$$TP_s = 2\eta \frac{\theta_2}{\theta_1} e_{\alpha\beta}^{\,2} + \frac{\theta_1}{2\eta (\theta_1 - \theta_2)} s_{\alpha\beta}^{\,2} \mp \frac{(1+\varepsilon)\mu}{\eta^2 (\theta_1 - \theta_2)^2} s_{\alpha\beta} s_{\beta\gamma} s_{\gamma\alpha} \qquad (3.19)$$

Let us consider another kind of particular case when nonlinear members with $\{\mathbf{hh}\}$ and $\{\mathbf{he}\}$ drop out of Eq. (3.11) for $\mathbf{h}$, i.e., when $b_{31} = b_3\lambda_1 / 4\mu$, $b_{21} = b_2\lambda_1 / 4\mu$ (by virtue of (3.12) this is equivalent to the requirement that $b_{22} = b_{32} = 0$). The equations of the model become

$$\left(\frac{\Delta \mathbf{h}}{\Delta t}\right)_{ij} + \frac{2\mu}{b_3} h_{ij} = \left(1 + \frac{b_2}{b_3}\right) e_{ij} \qquad (3.20)$$
$$\sigma_{ij} = \left(b_1 - \frac{b_2^{\,2}}{b_3}\right) e_{ij} + 2\mu h_{ij} + \lambda_1 \left(1 + \frac{b_2}{b_3}\right) h_{i\alpha} h_{\alpha j} +$$
$$+ \left(b_{11} - \frac{\lambda_1 b_2^{\,2}}{4\mu b_3}\right) (h_{i\alpha} e_{\alpha j} + e_{i\alpha} h_{\alpha j}) + b_{13} h_{\alpha\beta} e_{\alpha\beta} \delta_{ij}$$

In this case ($T = \text{const}$) the dissipation is

$$TP_s = \left(b_1 - \frac{b_2^{\,2}}{b_3}\right) e_{\alpha\beta}^{\,2} + \frac{4\mu^2}{b_3} h_{\alpha\beta}^{\,2} + \frac{2\mu\lambda_1}{b_3} h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} + 2\left(b_{11} - \frac{\lambda_1 b_2^{\,2}}{4\mu b_3}\right) h_{\alpha\beta} e_{\beta\gamma} e_{\gamma\alpha} \quad (3.21)$$

For $b_{12} = 0$, $b_{11} = \lambda_1 b_2^{\,2} / (4\mu b_3)$ Eqs. (3.20) reduce to equations for e_{ij} and $s_{ij} = \sigma_{ij} - (b_1 - b_2^{\,2}/b_3)\, e_{ij}$:

$$\left(\frac{\Delta s}{\Delta t}\right)_{ij} - \frac{\lambda_1}{2\mu}\left(1 + \frac{b_2}{b_3}\right)^2 (s_{i\alpha} e_{\alpha j} + e_{i\alpha} s_{\alpha j}) + \qquad (3.22)$$
$$+ \frac{2\mu}{b_3} s_{ij} + \frac{\lambda_1}{2\mu b_3}\left(1 + \frac{b_2}{b_3}\right) s_{i\alpha}\, s_{\alpha j} = \left(1 + \frac{b_2}{b_3}\right) e_{ij}$$

Here the case $b_2 = 0$, $s_{ij} = \sigma_{ij}$ corresponds to a model of Maxwell type ($\sigma_{ij} = \sigma_{ij}^{\,*}$), i.e., is described by equations of the same type.

Utilizing the concept of the Jaumann integral (see Section 5), we may pass from a differential to a functional description of the model equations.

Thus, Eqs. (3.20) are "solved" as follows (for simplicity we set $b_{12} = 0$, $b_{11} = \lambda_1 b^2{}_2/(4\mu b_3)$):

$$h_{ij}(t) = \left(1 + \frac{b_2}{b_3}\right) \int_{-\infty}^{t_*} [\mathrm{e}(t')]_{ij}$$

$$\sigma_{ij}(t) = \left(b_1 - \frac{b_2^2}{b_3}\right) e_{ij}(t) + 2\mu\left(1 + \frac{b_2}{b_3}\right) \int_{-\infty}^{t_*} [\mathrm{e}(t')]_{ij} + \tag{3.23}$$

$$+ \lambda_1 \left(1 + \frac{b_2}{b_3}\right)^2 \int_{-\infty}^{t_*} [\mathrm{e}(t')]_{i\alpha} \int_{-\infty}^{t_*} [\mathrm{e}(t'')]_{\alpha j}$$

Here the notation

$$\int_{-\infty}^{t_*} [\mathrm{e}(t')]_{ij} = \int_{-\infty}^{t} \varphi_{i\alpha}(t, t')\, e_{\alpha\beta}(t')\, \varphi_{\beta j}(t', t) \exp\left[-\frac{2\mu}{b_3}(t - t')\right] dt$$

has been introduced for the Jaumann integral, where ϕ is the matrizant, which satisfies Eq.

$$d\varphi_{ij}(t, t')/dt = \omega_{i\alpha}(t)\, \varphi_{\alpha j}(t, t'), \qquad \varphi_{ij}(t', t') = \delta_{ij}$$

in the case of Cartesian coordinates.

The model equations in the form (3.23) are analogous to the expansions of hereditary functionals in functional series utilized in the literature (see e.g. [1]). In addition to (3.23) it is not difficult to write an expression for the dissipative functional by using (3.21).

Up to now the exposition has relied on linear TIP relationships in the fluxes and forces. However, it is easy to see that taking account of the next terms, quadratic in the forces (fluxes), will not introduce additional difficulties but will just increase the arbitrariness of the coefficients. The new feature of the equations will just be the fact that terms of the $\{\mathbf{ee}\}_{ij}$ type will even occur in the equation for h_{ij} (compare with (3.11)). On the other hand, one "thermodynamic nonlinearity" in the linearized kinematic relationship of the quadratic internal energy and constant kinetic coefficients may result in a set of models with equations analogous to (3.11) and (3.13).

As an illustration, let us consider the case when the whole nonlinearity is due to the violation of the crossed symmetry between two phenomena (although the Onsager relationships are valid). Now, let (2.6) and (2.8) be replaced by (3.24)

$$\sigma_{ij} - \sigma_{ij}^e = b_1 e_{ij} + b_2 e_{ij}^p + d_1 \{e^p e^p\}_{ij} + d_2 e_{\alpha\beta}^{p2} \delta_{ij}, \quad \sigma_{ij}^e = 2\mu h_{ij} = b_2 e_{ij} + d_3 \{\mathrm{ee}\}_{ij} + b_3 e_{ij}^p$$

For simplicity, let us set $b_2 = 0$, in this case taking account of the nonlinear terms is particularly necessary. Retaining only terms of two orders, by using $e^p = e - \Delta\mathbf{h}/\Delta t$, Eq. (3.24) is easily transformed into

$$\left(\frac{\Delta \mathbf{h}}{\Delta t}\right)_{ij} + \frac{1}{\theta_1} h_{ij} = e_{ij} + \frac{d_3}{\mu\theta_1}\{\mathrm{ee}\}_{ij}$$

$$\sigma_{ij} = 2\eta \frac{\theta_2}{\theta_1} e_{ij} + 2\mu h_{ij} + \frac{2d_1}{\theta_1^2}\{\mathbf{hh}\}_{ij} + \frac{d_2}{\theta_1^2} h_{\alpha\beta}^2 \delta_{ij}, \tag{3.25}$$

where $\theta_1 - \theta_2 = \theta_1{}^2 \mu/\eta$ in this case.

With the same two order accuracy, we have for the dissipation (T = const)

$$TP_s = 2\eta \frac{\theta_2}{\theta_1} e_{\alpha\beta}^2 + \frac{2\mu}{\theta_1} h_{\alpha\beta}^2 + \frac{2d_1}{\theta_1^3} h_{\alpha\beta} h_{\beta\gamma} e_{\gamma\alpha} - \frac{2d_3}{\theta_1} e_{\alpha\beta} e_{\beta\gamma} h_{\gamma\alpha}$$

In concluding the Section, let us estimate the role of the nonlinearity in the kinematic relationship. When we may limit oneself to terms of three orders in the kinematic relationship, while the rest of the relationships are considered linear, the governing equations will become

$$\left(\frac{\Delta s}{\Delta t}\right)_{ij} + \frac{2\mu}{b_3} s_{ij} + \frac{b_2 - 2b_3}{12\mu(b_2+b_3)} (s_{i\alpha} s_{\alpha\beta} e_{\beta j} - 2 s_{i\alpha} e_{\alpha\beta} s_{\beta j} +$$
$$+ e_{i\alpha} s_{\alpha\beta} s_{\beta j}) = \left(1 + \frac{b_2}{b_3}\right) e_{ij}, \qquad s_{ij} = \sigma_{ij} - \left(b_1 - \frac{b_2^2}{b_3}\right) e_{ij} \tag{3.26}$$

A comparison of predictions of a model with complete kinematic nonlinearity and a model with a "linearized" kinematic relationship will be made later in a simple shear flow example. We consider the medium incompressible, and all the remaining relationships linear (for example, $\sigma_{ij}^{\circ} = 2\mu h_{ij}$).

The problem of stationary shear flow with shear velocity $\gamma^{\cdot}$ hence reduces to the solution of the matrix Eqs. (see (1.11))

$$\frac{2}{\theta_1}\mathbf{h} - 2\frac{b_2}{b_3}\mathbf{e} = \exp(-\mathbf{h})[\mathbf{e} - \boldsymbol{\omega}]\exp(\mathbf{h}) + \exp(\mathbf{h})[\mathbf{e} + \boldsymbol{\omega}]\exp(-\mathbf{h})$$
$$\boldsymbol{\sigma} = 2\eta\frac{\theta_2}{\theta_1}\mathbf{e} + 2\mu\left(1 + \frac{b_2}{b_3}\right)\mathbf{h} \tag{3.27}$$

In the case under consideration the matrices $\mathbf{e}$, $\boldsymbol{\omega}$, $\mathbf{h}$ are

$$\mathbf{e} = \frac{1}{2}\gamma^{\cdot}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \boldsymbol{\omega} = \frac{1}{2}\gamma^{\cdot}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix}h_{11} & h_{12}\\ h_{12} & -h_{11}\end{pmatrix}$$

In order to solve the first matrix equation for the matrix $\mathbf{h}$, let us perform a similarity transformation such that the matrix $\mathbf{h}$ would reduce to the diagonal form

$$\mathbf{q}^{-1}\mathbf{h}\mathbf{q} = \begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}$$

In the considered case $h_{\alpha\alpha} = 0$, $h_{\alpha\beta}^2 = 2(h_{11}^2 + h_{12}^2) = 2a^2$, $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = 0$. Introducing new unknowns a and ϕ in place of h_{11}, h_{12} by setting $h_{11} = a\cos\phi$ $h_{12} = a\sin\phi$, it is easy to see that the matrix $\mathbf{q}$ is hence an orthogonal rotation matrix

$$q = \begin{pmatrix}\cos{}^1/_2\varphi & -\sin{}^1/_2\varphi\\ \sin{}^1/_2\varphi & \cos{}^1/_2\varphi\end{pmatrix}$$

and the matrix equation reduces to two transcendental Eqs.

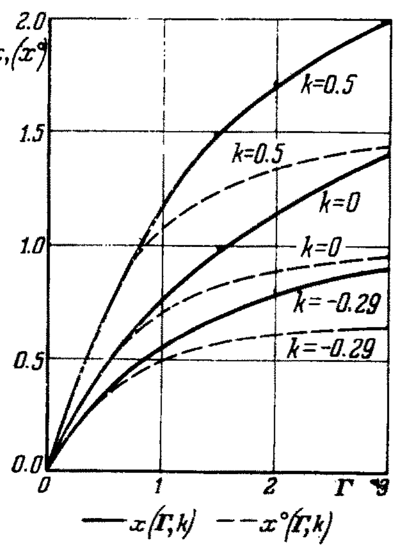

Fig. 1

$$2a = \Gamma(1+k)\sin\varphi$$
$$\operatorname{sh} 2a = (\operatorname{ch} 2a + k)\cos\varphi \tag{3.28}$$

Here $\Gamma = \theta_1\gamma^{\cdot}$ and $k = b_2/b_3$ are nondimensional parameters. The parameter k is expressed in terms of the dimensional constants η, μ, θ_1, θ_2:

$$(1+k)^2 = \frac{\eta}{\mu\theta_1}\left(1 - \frac{\theta_2}{\theta_1}\right)$$

The components of the elastic deformation and stress tensors may be expressed in terms of the parameter a as follows:

$$h_{11} = \frac{a\operatorname{sh}2a}{\operatorname{ch}2a + k}, \qquad h_{12} = \frac{2a^2}{\Gamma(1+k)}$$
$$\sigma_{11} = 2\mu(1+k)\frac{a\operatorname{sh}2a}{\operatorname{ch}2a + k}$$
$$s_{12} \equiv \sigma_{12} - \frac{\eta\theta_2}{\theta_1^2}\Gamma = 4\mu a^2\Gamma^{-1} \tag{3.29}$$

The dependence of the parameter a on Γ and k is found from (3.28).

The linearized kinematic relationship for the problem being solved is

$$\boldsymbol{\omega}\mathbf{h}^{\circ} - \mathbf{h}^{\circ}\boldsymbol{\omega} + \mathbf{h}^{\circ}/\theta_1 = (1+k)\mathbf{e}$$

and in this case it is easy to write an explicit expression for a°, and hence, for σ°

$$a^\circ = \frac{(1+k)\Gamma}{2\sqrt{1+\Gamma^2}}, \qquad \sigma_{11}^\circ = \frac{\mu(1+k)^2\Gamma^2}{1+\Gamma^2}, \qquad s_{12}^\circ = \frac{\sigma_{11}^\circ}{\Gamma} \tag{3.30}$$

Comparison of the solution (3.29), (3.30) is made graphically. Pictured in Figs. 1-3 are de-

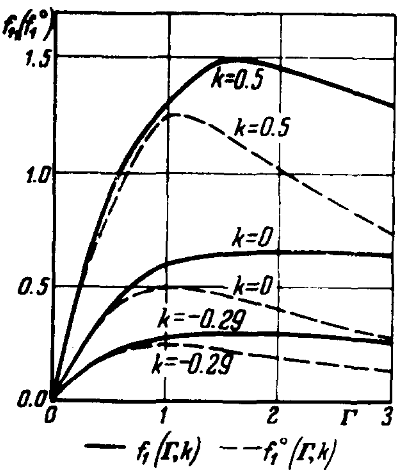

Fig. 2

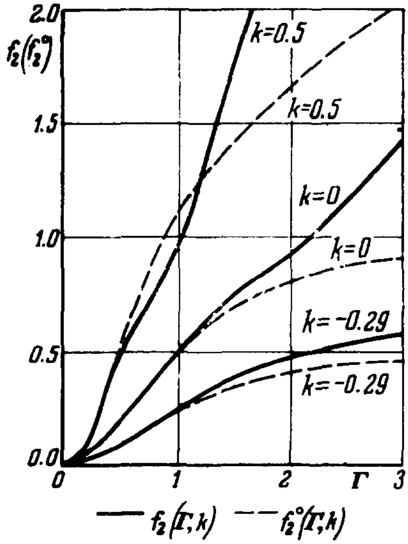

Fig. 3

pendences of the nondimensional quantities $x = 2a$, $f_1 = s_{12}/\mu$, $f_2 = \sigma_{11}/\mu$ and the corresponding quantities x°, f_1°, f_2° on the parameter Γ for several numerical values of the parameter k. It is seen from the graphs that larger elastic deformations in the fluid correspond to larger values of the parameter k (for the same Γ); the functions f_1 and f_1° have one maximum each (in order for $\sigma_{12}^\circ(\Gamma)$ to be monotonous here it is necessary that $\theta_2 > \theta_1/9$), down to $\Gamma \approx 1$ the theory with the linearized kinematic relationship yields an error not exceeding 10 %, however the error grows rapidly for large values of Γ.

4. Model with many relaxation times: Viscoelastic spectra. Let us consider an element of fluid which consists of N subsystems. The elastic deformation of the k-th subsystem is described by the tensor $\mathbf{h}^{(k)}$ and the irreversible strain rate by $\mathbf{e}^{p(k)}$. The preceding analysis can be extended to this more general case exactly as is done in linear viscoelasticity [4 and 21]. A sufficiently simple case is considered below. We write the internal energy to the accuracy of cubic terms in $\mathbf{h}$ and the linearized kinematic relationship as

$$\rho_0 u = \rho_0 u_0 + \sum_k \mu_k \mathbf{h}^{(k)} \cdot\cdot\, \mathbf{h}^{(k)} + \sum_{k,\,l} \lambda_{kl} \mathbf{h}^{(k)} \cdot\cdot\, \mathbf{h}^{(l)} \cdot\cdot\, \mathbf{h}^{(l)} \tag{4.1}$$

$$\frac{\Delta \mathbf{h}^{(k)}}{\Delta t} + \mathbf{e}^{p(k)} = \mathbf{e}, \qquad \lambda_{kl} = \lambda_{lk}, \qquad \mathbf{h} \cdot\cdot\, \mathbf{h} = h_{i\alpha}^2 \tag{4.2}$$

Such expressions may be obtained from the more general case by reduction to normal coordinates (see [4 and 21]).

The expression for entropy production (T = const) can be written as

$$TP_s = \Big[\boldsymbol{\sigma} - \sum_k \boldsymbol{\sigma}^{e(k)}\Big] \cdot\cdot\, \mathbf{e} + \sum_k \boldsymbol{\sigma}^{e(k)} \cdot\cdot\, \mathbf{e}^{p(k)} \tag{4.3}$$

$$\boldsymbol{\sigma}^{e(k)} = 2\mu_k \mathbf{h}^{(k)} + \sum_l \lambda_{kl} \left[2\mathbf{h}^{(k)}\mathbf{h}^{(l)} + \mathbf{h}^{(l)}\mathbf{h}^{(l)}\right] \tag{4.4}$$

Henceforth, the simple case of a Maxwell type model is considered when

$$\sigma = \sum_k \sigma^{e\,(k)}, \qquad \mathbf{h}^{(k)} = \theta_k \mathbf{e}^{p\,(k)}, \qquad \theta_k > 0 \tag{4.5}$$

Eqs. (4.2) and (4.5) may be solved for $\mathbf{h}^{(k)}$ by using Jaumann integration (Section 5 and [22])

$$h_{ij}^{(k)}(t) = \int_{-\infty}^{t} {}^{*} \exp\left(-\frac{t-t'}{\theta_k}\right)[\mathbf{e}(t')]_{ij}$$

Substituting $\mathbf{h}^{(k)}$ into the expression for $\sigma^{e(k)}$ and taking account of (4.5) we obtain

$$\sigma_{ij}(t) = 2\int_{-\infty}^{t} {}^{*} \psi_0(t-t')[\mathbf{e}(t')]_{ij} + 2\int_{-\infty}^{t} {}^{*} \int_{-\infty}^{t} {}^{*} \psi_1(t-t',\, t-t'')[\mathbf{e}(t')]_{i\alpha}[\mathbf{e}(t'')]_{\alpha j} \tag{4.6}$$

We have here introduced the notation

$$\psi_0(t) = \sum_{k=1}^{N} \mu_k \exp\left(-\frac{t}{\theta_k}\right)$$

$$\psi_1(t, t'') = \sum_{k,l=1}^{N} \lambda_{kl}\left[\exp\left(-\frac{t'}{\theta_k} - \frac{t''}{\theta_l}\right) + \frac{1}{2}\exp\left(-\frac{t'+t''}{\theta_k}\right)\right]$$

In order to pass from the discrete to the continuous relaxation-time spectrum, it is only necessary to replace the summation by integration, and the coefficients μ_k, λ_{lk} by the spectral relaxation functions $\mu(\theta)$, $\lambda(\theta_1, \theta_2) = \lambda(\theta_2, \theta_1)$.

Let us consider the behavior of such a model medium for simple kinds of flows.

Quasistationary Couette Flow Mode with Instantaneous Inclusion of the Strain Rate. In this case

$$\mathbf{e} = \frac{1}{2}\gamma^{\cdot} H(t)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \omega = \frac{1}{2}\gamma^{\cdot} H(t)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Here $\gamma^{\cdot}$ is a constant shear rate, $H(t)$ the Heaviside unit function, which equals zero for $t < 0$ and 1 for $t > 0$.

The matrizant $\phi(t, t')$ which should satisfy Eq.

$$\frac{\partial}{\partial t}\varphi(t, t') = -\omega\varphi(t, t'), \qquad \varphi(t', t') = \mathrm{I}$$

is found easily to have the form

$$\varphi(t, t') = \begin{pmatrix} \cos{}^1/_2\gamma^{\cdot}(t-t') & -\sin{}^1/_2\gamma^{\cdot}(t-t') & 0\\ \sin{}^1/_2\gamma^{\cdot}(t-t') & \cos{}^1/_2\gamma^{\cdot}(t-t') & 0\\ 0 & 0 & 1\end{pmatrix}$$

and is a matrix of rotation through the angle $\frac{1}{2}\gamma^{\cdot}(t - t')$.

The stress tensor components (less the pressure) may be expressed in terms of the spectral functions as follows:

$$\sigma_{12} = \int_0^{\infty} \mu(\theta)\frac{\theta\gamma^{\cdot}u(t,\theta)}{1+\theta^2\gamma^{\cdot 2}}d\theta, \qquad \sigma_{22} - \sigma_{11} = -2\int_0^{\infty}\mu(\theta)\frac{\theta^2\gamma^{\cdot 2}v(t,\theta)}{1+\theta^2\gamma^{\cdot 2}}d\theta \tag{4.7}$$

$$\sigma_{11} = \int_0^{\infty}\mu(\theta)\frac{\theta^2\gamma^{\cdot 2}v(t,\theta)}{1+\theta^2\gamma^{\cdot 2}}d\theta + \frac{1}{4}\int_0^{\infty}\beta(\theta)\frac{\theta^2\gamma^{\cdot 2}[\theta^2\gamma^{\cdot 2}v^2(t,\theta) + u^2(t,\theta)]}{(1+\theta^2\gamma^{\cdot 2})^2}d\theta +$$

$$+\frac{1}{2}\int_0^{\infty}\int_0^{\infty}\lambda(\theta_1,\theta_2)\frac{\theta_1\theta_2\gamma^{\cdot 2}[\theta_1\theta_2\gamma^{\cdot 2}v(t,\theta_1)\,v(t,\theta_2) + u(t,\theta_1)\,u(t,\theta_2)]}{(1+\theta_1^2\gamma^{\cdot 2})(1+\theta_2^2\gamma^{\cdot 2})}d\theta_1\,d\theta_2$$

where we have introduced the notation

$$\beta(\theta) = \int_0^\infty \lambda(\theta, \tau)\, d\tau \tag{4.8}$$

$$u(t, \theta) = 1 - \exp\left(-\frac{t}{\theta}\right)(\cos\gamma^{\cdot} t - \gamma^{\cdot}\theta \sin\gamma^{\cdot} t)$$

$$v(t, \theta) = 1 - \exp\left(-\frac{t}{\theta}\right)\left(\cos\gamma^{\cdot} t + \frac{1}{\gamma^{\cdot}\theta}\sin\gamma^{\cdot} t\right)$$

It follows from (4.7) and (4.8) that both the normal and the tangential stresses have damped oscillations upon emergence into the steady flow regime. The first maximum of the tangential stress is the largest and has the value:

$$\max \sigma_{12} = \int_0^\infty \mu(\theta) \frac{\theta\gamma^{\cdot}}{1 + \theta^2\gamma^{\cdot 2}}\left[1 + \exp\left(-\frac{\pi}{2\theta\gamma^{\cdot}}\right)\right] d\theta$$

Let us note that apparently such oscillations were observed experimentally in the polymer rheology laboratory of the Institute of Petrochemical Synthesis AN SSSR by G.V. Vinogradov and A.Ia. Malkin.

Formulas (4.7) and (4.8) simplify in the limit $t \to \infty$

$$\sigma_{12} = \int_0^\infty f(\theta) \frac{\theta\gamma^{\cdot}}{1 + \theta^2\gamma^{\cdot 2}}\, d\theta, \quad \sigma_{11} - \sigma_{22} = 2\int_0^\infty \mu(\theta) \frac{\theta^2\gamma^{\cdot 2}}{1 + \theta^2\gamma^{\cdot 2}}\, d\theta \tag{4.9}$$

$$\sigma_{11} = \int_0^\infty \mu(\theta) \frac{\theta^2\gamma^{\cdot 2}}{1 + \theta^2\gamma^{\cdot 2}}\, d\theta + \frac{1}{4}\int_0^\infty \beta(\theta) \frac{\theta^2\gamma^{\cdot 2}}{1 + \theta^2\gamma^{\cdot 2}}\, d\theta +$$

$$+ \frac{1}{2}\int_0^\infty\int_0^\infty \lambda(\theta_1, \theta_2) \frac{\theta_1\theta_2\gamma^{\cdot 2}(1 + \theta_1\theta_2\gamma^{\cdot 2})}{(1 + \theta_1^2\gamma^{\cdot 2})(1 + \theta_2^2\gamma^{\cdot 2})}\, d\theta_1\, d\theta_2$$

The dependence of the effective viscosity $\eta^\circ(\gamma^{\cdot}) = \sigma_{12}(\gamma^{\cdot})/\gamma^{\cdot}$ agrees with the dynamic viscosity $\eta'(\omega)$ (determined in the linear theory of viscoelasticity) for $\gamma^{\cdot} \to \omega$. The quantities σ_{11} and σ_{22} in (4.9) are not equal, and are quadratic for small $\gamma^{\cdot}$ as $\gamma^{\cdot} \to 0$. It is interesting to note that the anharmonic terms do not yield a contribution to the tangential stresses in simple one-dimensional flows of the considered medium. The difference $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ agrees with the real part of the dynamic modulus $G'(\omega)$ as $\gamma^{\cdot} \to \omega$. The agreement between η° and η', $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and G' has been discussed repeatedly in the rheological literature (see [19, 20 and 23], for example).

Let us consider yet another simple experiment on whose basis the functions $\sigma_{11}(\gamma^{\cdot})$ and $\sigma_{22}(\gamma^{\cdot})$ may be estimated. Let a viscoelastic fluid move stationarily in the narrow gap of a rotating cone-plane device customarily utilized for rheological investigations. Then, as is easy to show for this case, the tangential and normal stress distribution is

$$p_{\varphi\varphi} = -p + \sigma_{11}(\gamma^{\cdot}), \quad p_{\theta\theta} = -p + \sigma_{22}(\gamma^{\cdot}), \quad p_{rr} = -p, \quad p_{\theta\varphi} = \sigma_{12}(\gamma^{\cdot}), \quad p_{r\varphi} = p_{r\theta} = 0 \tag{4.10}$$

Here the quantities $\sigma_{ij}(\gamma^{\cdot})$ are defined in (4.9). From the equilibrium equations we have a simple equation for the distribution of the isotropic pressure over the radius of the device (the fact that the angular gap is small is used here)

$$\frac{\partial p}{\partial r} \approx \frac{1}{r}(2p_{rr} - p_{\varphi\varphi} - p_{\theta\theta}) = -\frac{\sigma_{11} + \sigma_{22}}{r}$$

Integrating this equation while taking into account that there is a free surface for $r = R$ on which $p_{rr} = 0$, we obtain

$$p = (\sigma_{11} + \sigma_{22}) \ln(R / r) \tag{4.11}$$

Evaluating the axial pressure according to (4.10) and (4.11), we find

$$Q = 2\pi \int_0^R p_{\theta\theta} r dr = -\frac{\pi}{2} R^2 (\sigma_{11} - \sigma_{22}) \tag{4.12}$$

According to [24], results of experiments show that $\sigma_{11} > \sigma_{22}$ and $\sigma_{11} \sim \sigma_{22}$. The latter inequality corresponds to the constraint $\lambda(\theta_1, \theta_2) > 0$ on the binary relaxation function λ.

Let us note yet another curious fact. The results of many normal stress tests (including those in [24]) convincingly show a logarithmic pressure distribution over the radius in the cone-plane device. These tests thereby (see (4.11)) show the inapplicability of rheological models in which the two-dimensional tensor σ_{ij} is a deviator in simple shear flow, for the description of normal stresses (such a situation arises particularly if anharmonic terms in (4.1) are neglected).

Let us now examine steady flow with simple tension of a film of viscoelastic fluid.

Let there be the following velocity distribution in the liquid film (x is the motion direction, y the transverse coordinate)

$$v_x = \gamma^{\cdot} x, \quad v_y = \gamma^{\cdot} y \quad (\gamma^{\cdot} = \text{const})$$

The strain rate tensor has the form

$$e = \gamma^{\cdot} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.13}$$

In this case the vorticity tensor is evidently $\omega = 0$ and the matrizant is $\phi_{ij} = \delta_{ij}$. On the basis of (4.6) and (4.13) we have

$$\sigma = 2\eta e + 2\nu e^2, \quad \eta = \int_0^\infty \psi_0(t)\, dt, \quad \nu = \int_0^\infty \int_0^\infty \psi_1(t', t'')\, dt' dt'' \tag{4.14}$$

Formula (4.14) shows that Trouton viscosity, defined by means of σ_{xx}, grows with increasing $\gamma^{\cdot}$, which also corresponds qualitatively with experiment.

By virtue of the evident approximate nature of the obtained equations, the description of the normal stresses, non-Newtonian viscosity, etc., by such governing equations as in Sections 3 and 4, can claim only qualitative agreement with experiment although they are good enough for some materials (for instance, polymer solutions, see [19, 20, 23, and 24]).

Moreover, the viscoelasticity theory constructed in this Section is based on the relationships (4.1) and (4.2), which are constrained by sufficient smallness of the elastic deformations. If the value of the mean elastic deformation is characterized by the parameter $\Gamma = \langle\theta\rangle \gamma_0^{\cdot}$, where $\langle\theta\rangle$ is some relaxation time averaged over the spectrum, and $\gamma_0^{\cdot}$ is the characteristic shear rate, then the domain of applicability of the constructed theory is bounded by the inequality $\Gamma < 1$. Since the values of $\langle\theta\rangle$ may be sufficiently large for polymer melts and concentrated solutions, the domain of applicability of the theory actually turns out to be bounded by values of sufficiently small $\gamma^{\cdot}$. Reversible ruptures in the structure, thixotropy [25], may, in addition to the geometric nonlinearity noted above, give a substantial contribution to the phenomenon of the viscosity anomaly in viscoelastic media of the polymer melt type. Taking account of thixotropic effects in viscoelastic media can be done within the framework of formal thermodynamics of irreversible processes by following the ideas of [25], however, this is outside the scope of the present paper.

5. Appendix. On the Jaumann integral. Let x^k be an arbitrary fixed coordinate system; let $A_k{}^i$, $B_k{}^i$ be some second rank tensors with mixed indices obtained from the symmetric tensors A_{ik}, B_{ik} by the operation of raising the index. Let the tensor $B_k{}^i$ be given. Let us examine the equation in $A_k{}^i$:

$$\left(\frac{\Delta A}{\Delta t}\right)_k^i = \frac{\partial A_k^i}{\partial t} + v^\alpha \nabla_\alpha A_k^i + \omega_\alpha^i A_k^\alpha - A_\alpha^i \omega_k^\alpha = B_k^i, \ 2\omega_k^i = \nabla_k v^i - \nabla^i v_k \tag{5.1}$$

Here $\omega_k{}^i$ is the vorticity tensor, ∇_α the operation of covariant differentiation. Let us find the solution of this equation for a given velocity vector $v^i = v^i(x^k, t)$ and some initial condition $A_k^{\,i} = C_k^{\,i}$ at $t = t_0$. Here $C_k^{\,i}(x^m)$ is some tensor independent of the time t.

Such a problem of inverting the Jaumann derivative (i.e., the problem of constructing the Jaumann integral) was considered schematically in [22].

The complete solution of (5.1) will be considered here by using a generalization of the method in]22] and the solution of an analogous problem in a frozen Lagrangean system of coordinates ξ^k. Let us consider the solution of the problem in a fixed coordinate system.

Introducing the notation

$$\frac{d^*}{dt} = \frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial x^\alpha}, \quad \omega^{*i}_{\cdot j} = \omega^{i}_{\cdot j} + v^\alpha \Gamma^i_{\alpha j}$$

where $\Gamma_{\alpha j}^{\,i}$ is the Christoffel symbol, and the extensive $\omega^* = \|\omega^{*\cdot\cdot}_{ij}\|$ coincides with ω only in a Cartesian coordinate system (evidently, d^*/dt is the nontensor time derivative), we write (5.1) in the matrix form

$$\frac{\Delta \mathbf{A}}{\Delta t_J} = \frac{d^*\mathbf{A}}{dt} + \omega^*\mathbf{A} - \mathbf{A}\omega^* = \mathbf{B} \tag{5.2}$$

Let us examine the solution of the auxiliary matrix equation (5.3)

$$d^*/dt\ \varphi(t,\ t_0;\ x^k) = -\omega^*(t,\ x^k)\,\varphi(t,\ t_0;\ x^k),\ \varphi(t_0,\ t_0;\ x^k) = \|\varphi_{i\cdot}^{\cdot j}(t_0,\ t_0;\ x^k)\| = \mathbf{I} = \|\delta_{i\cdot}^{\cdot j}\|$$

Following [10], let us introduce the "displacement function" $x'^k(x^i,\ t,\ t')$ which describes the position of a continuum point with the fixed Lagrangean coordinate ξ^k at the time t' under the condition that the point occupied the position x^k at time t. Evidently the displacement functions are solutions of the Cauchy problem [10]

$$\frac{\partial x'^k}{\partial t} + v^\alpha \frac{\partial x'^k}{\partial x^\alpha} = 0, \qquad x'^k(x^i,\ t,\ t')\,|_{t'=t} = x^k \tag{5.4}$$

From (5.4) it is easy to note that

$$x''^k(x'^i,\ t',\ t'') = x''^k(x^i,\ t,\ t'')$$

An iterative solution of (5.3) is

$$\varphi(t,\ t_0;\ x^k) = I - \int_{t_0}^{t} \omega^*(t',\ x'^k)\,dt' + \int_{t_0}^{t}\int_{t_0}^{t} \omega^*(t',\ x'^k)\,\omega^*(t'',\ x''^k)\,dt'dt'' + \dots \tag{5.5}$$

The quantity ϕ is customarily called the matrizant of the matrix differential equation. Let us note that according to (5.5), ϕ depends on x^k only in terms of ω^*, where ϕ is generally a nonsymmetric functional of ω^*. We shall henceforth omit the argument x^k or the functional argument ω^* in the notation for ϕ.

The order of the variables t, t_0 in the notation of the matrizant ϕ is quite essential since t denotes current time, and t_0 is the lower limit of the integration in (5.5) corresponding to some reference point. The properties of the matrizant

$$\varphi(t,\ t_0)\,\varphi(t_0,\ t_1) = \varphi(t,\ t_1), \qquad \varphi(t,\ t_0)\,\varphi(t_0,\ t) = \mathbf{I} \tag{5.6}$$

are easily proved by using (5.5).

From (5.3) and the second property of (5.6) we easily deduce

$$d^*/dt\,\varphi(t_0,\ t) = \varphi(t_0,\ t)\,\omega^*, \qquad \varphi(t_0,\ t_0) = \mathbf{I} \tag{5.7}$$

The solution of (5.2) with the aid of ϕ may be written as [22]

$$\mathbf{A} = \mathbf{C} + \int_{t_0}^{t} \varphi(t,\ t';\ \omega^*)\,\mathbf{B}(t',\ \mathbf{x}')\,\varphi(t',\ t;\ \omega^*)\,dt' \equiv \mathbf{C} + \int_{t_0}^{t}{}^{*}\,[\mathbf{B}(t');\ \omega^*] \tag{5.8}$$

In particular

$$\operatorname{Sp}\mathbf{A} = \operatorname{Sp}\mathbf{C} + \int_{t_0}^{t} \operatorname{Sp}\mathbf{B}(t',\ x')\,dt', \qquad \operatorname{Sp}\mathbf{A} \equiv A_k^k$$

It is interesting to note that despite the nontensor nature of the matrizant ϕ the desired quantity **A** in (5.8) is of tensor nature because of the tensor nature of the Jaumann derivative and of the right side **B** in (5.2).

It is not difficult to show that the customary integration by parts formula with the scalar function $f(t)$ holds for the integral in (5.8):

$$\int_{t_0}^{t}{}^{*}\left[\frac{\Delta \mathbf{A}}{\Delta t} f\right] = f\varphi \frac{\Delta \mathbf{A}}{\Delta t}\varphi^{-1}\Big|_{t_0}^{t} - \int_{t_0}^{t}\left[\mathbf{A}\frac{df}{dt}\right] \tag{5.9}$$

Let us consider an example. Let it be required to find the tensor $\sigma(x^k, t)$ by means of the known tensor $\mathbf{e}$ from Eq.

$$\theta_1 \frac{\Delta\sigma}{\Delta t} + \sigma = 2\eta\left(\theta_2 \frac{\Delta \mathbf{e}}{\Delta t} + \mathbf{e}\right) \equiv \mathbf{B}, \quad \sigma \quad (x^k, t)\underset{t\to-\infty}{\to} 0 \tag{5.10}$$

From (5.10) we have

$$\sigma = \frac{1}{\theta_1}\int_{-\infty}^{t}{}^{*} \exp\left(-\frac{t-t'}{\theta_1}\right)[\mathbf{B}(t')]$$

Substituting its value from (5.10) for the tensor **B** in this expression, and integrating while taking account of all the tensors vanishing as $t \to -\infty$, we obtain

$$\sigma = 2\eta\frac{\theta_2}{\theta_1}\mathbf{e} + 2\eta\frac{\theta_1-\theta_2}{\theta_1^2}\int_{-\infty}^{t}{}^{*}\exp\left(-\frac{t-t'}{\theta_1}\right)[\mathbf{e}(t');\ \omega^*]$$

Now, let us consider the determination of the Jaumann derivative and the Jaumann integral in a convective frozen coordinate system ξ^k. Let $a_{ij}(\xi^k, t)$ and $a^{ij}(\xi^k, t)$ be components of the symmetric tensor **A** in the system ξ^k. Let us introduce the three convective derivatives

$$b_{ij}^{(1)} = \frac{Da_{ij}}{Dt}, \quad b^{(2)ij} = \frac{Da^{ij}}{Dt}, \quad b^{(3)i}{}_{\cdot j} = \frac{Da^{i}{}_{\cdot j}}{Dt}, \quad \frac{D}{Dt} \equiv \frac{\partial}{\partial t}\Big|_{\xi^k}, \quad \mathbf{B}^{(k)} := b_{ij}^{(k)}\partial_1^i\partial_1^j$$

Here ∂_{1i} is the moving Lagrange basis in the deformed space (Section 1). In general tensors $\mathbf{B}^{(k)}$ are all distinct, which is associated with the fact that generally

$$\frac{D}{Dt}g_{ik}^{(1)} = 2e_{ik}, \quad \frac{D}{Dt}g^{(1)ik} = -2e^{ik} \tag{5.11}$$

are nonzero.

Here $\mathbf{e}_{ik}$ are the components of the strain rate tensor with respect to the basis ∂_1^i with the fundamental tensor $g_{ik}^{(1)}$. Let us note that the tensors $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ are symmetric, while the tensor $\mathbf{B}^{(3)}$ is asymmetric.

Now let us consider the symmetric tensor with mixed components

$$b_k^i = \frac{1}{2}\left(b^{(1)i}_{k} + b^{(2)i}_{k}\right) = \frac{1}{2}\left(g^{(1)i\alpha}\frac{Da_{\alpha k}}{Dt} + g_{k\alpha}^{(1)}\frac{Da^{\alpha i}}{Dt}\right) \tag{5.12}$$

By virtue of (5.11) we may represent (5.12) in terms of components of the tensor **A** with a different arrangement of the indices

$$b_k^i = \frac{Da_k^i}{Dt} + a_k^\alpha e_\alpha^i - e_k^\alpha a_\alpha^i \equiv \frac{D'a_k^i}{Dt} \tag{5.13}$$

$$b^{ik} = \frac{Da^{ik}}{Dt} + a^{i\alpha}e_\alpha^k + e_\beta^i a^{\beta k} \equiv \frac{D'a^{ik}}{Dt}, \qquad b_{ik} = \frac{Da_{ik}}{Dt} - e_i^\alpha a_{\alpha k} - a_{i\alpha}e_k^\alpha \equiv \frac{D'a_{ik}}{Dt}$$

Formulas (5.12) and (5.13) define the Jaumann derivative D'/Dt of the symmetric tensor **A** with respect to the convective basis ∂_1^i. Completely analogously, the Jaumann derivative of the nonsymmetric tensor could be defined with respect to the basis ∂_1^i. The fundamental properties of the Jaumann derivative

$$\frac{D'g^{(1)}_{ik}}{Dt}=0, \qquad g^{(1)i\alpha}\frac{D'a_{\alpha k}}{Dt}=\frac{D'a^{i\cdot}_{\cdot k}}{Dt}$$

are easily proved.

Let us consider the question of inverting the operation of Jaumann differentiation in the convective coordinate system with basis $\partial_1{}^i$. Let us write the first equality in (5.13) in matrix (tensor) notation

$$\frac{D'\mathbf{a}}{Dt}\equiv\frac{D\mathbf{a}}{Dt}+\mathbf{ea}-\mathbf{ae}=\mathbf{b} \tag{5.14}$$

Let us find the solution of (5.14), the tensor **a**, which vanishes at time t_0 by assuming the tensors **b** and **e** known.

Let us introduce the matrizant $\psi(t,\ t_0;\ \xi^k)=\|\psi^{i\cdot}{}_{\cdot j}\ (t,\ t_0;\ \xi^k)\|$ as the solution of a problem with the initial data

$$D\psi\,/\,Dt=-\,\mathbf{e}\psi, \qquad \psi\,(t_0,\ t_0;\ \xi^k)=\mathbf{I}=\|\,\delta^{i\cdot}_{\cdot j}\,\| \tag{5.15}$$

From (5.15) it is easy to see that ψ is generally a nonsymmetric tensor. The iteration solution for ψ is

$$\psi\,(t,\ t_0;\ \xi^k)=\mathbf{I}-\int_{t_0}^{t}\mathbf{e}\,(t',\ \xi^k)\,dt'+\int_{t_0}^{t}dt'\int_{t_0}^{t'}\mathbf{e}\,(t',\ \xi^k)\,\mathbf{e}\,(t'',\ \xi^k)\,dt''-\ldots \tag{5.16}$$

Moreover, the tensor-matrizant ψ possesses all the properties of the ordinary matrizant since the matrix Eq. (5.15) is a system of ordinary differential equations. In particular, the properties (5.6) are satisfied for ψ, where the equation for $\psi\,(t_0,\ t;\ \xi^k)$ is

$$D\,/\,Dt\,\psi\,(t,\ t_0;\ \xi^k)=\psi\,(t_0,\ t;\ \xi^k)\,\mathbf{e}\,(t,\ \xi^k),\ \ \psi\,(t_0,\ t_0;\ \xi^k)=\mathbf{I}$$

It follows from (5.16) that ψ is a functional of **e** and depends on ξ^k only in terms of **e**; hence it is natural to write $\psi(t,\ t_0;\ \mathbf{e})$. As above, it is easy to obtain the solution of (5.14) in the form

$$\mathbf{a}=\int_{t_0}^{t}\psi(t,\ t';\ \mathbf{e})\,\mathbf{b}\,(\xi^k,\ t')\,\psi\,(t',\ t;\ \mathbf{e})\,dt'\equiv\int_{t_0}^{t}{}^{*}\,[\mathbf{b}\,(t');\ \mathbf{e}] \tag{5.17}$$

Another tensor $\psi^{\cdot}{}_{k}{}^{i}{}_{\cdot}(t,\ t_0;\ \mathbf{e})$ could be introduced in place of the tensor $\psi^{i\cdot}{}_{\cdot k}\,(t,\ t_0;\ e)$ however it is easy to see that

$$\psi_{k\cdot}^{\cdot i}=[\psi^{k\cdot}_{\cdot i}]^T$$

The symbol T here denotes transposition.

Just as had been done above, it is easy to find the solution of the tensor equation

$$D'\mathbf{a}\,/\,Dt+\lambda\mathbf{a}=\mathbf{b},\ \ \mathbf{a}\,|_{t=t_0}=0$$

(**b** is a given tensor, λ a scalar constant) in the frozen ξ^k coordinate system. The solution of this equation will be

$$\mathbf{a}=\int_{t_0}^{t}{}^{*}\exp\,(-\,\lambda\,(t-t'))\,[\mathbf{b}\,(t');\ \mathbf{e}]$$

Transforming in (5.17) from the ξ^k coordinate system to the fixed x^k coordinate system, according to the rules set up in [10], we obtain

$$A^i_k=\int_{t_0}^{t}\frac{\partial x^i}{\partial x'^m}\,\psi^{m\cdot}_{\cdot\alpha}\,(t,\ t';\ \mathbf{e})\,B^\alpha_\beta\,(t',\ x')\,\psi^{\beta\cdot}_{\cdot\gamma}\,(t',\ \ t,\ \mathbf{e})\ \frac{\partial x'^\gamma}{\partial x^k}\,dt' \tag{5.18}$$

Here $A_k{}^i$, $B_k{}^i$, $\psi^{i\cdot}{}_{\cdot k}$, $e_k{}^i$ are tensor components in the fixed x^k coordinate system; the quantities x'^k are displacement functions defined by the solution of the problem (5.4).

The tensor-matrizant $\psi^{\beta}_{\cdot\alpha}{}^{\cdot}(t,\ t_0,\ \mathbf{e})$ is defined by the expression

$$\psi^{i}_{\cdot j} = \delta^{i}_{j} - \int\limits_{t_0}^{t} \frac{\partial x^{i}}{\partial x'^{\alpha}} e^{\alpha}_{\beta}(x', t') \frac{\partial x'^{\beta}}{\partial x^{j}} dt' + \int\limits_{t_0}^{t} dt' \int\limits_{t_0}^{t'} \frac{\partial x^{i}}{\partial x'^{\alpha}} e^{\alpha}_{\beta}(x', t') \frac{\partial x'^{\beta}}{\partial x''^{\gamma}} e^{\gamma}_{\nu}(x'', t'') \frac{\partial x''^{\nu}}{\partial x^{j}} dt'' - \ldots \qquad (5.19)$$

In combination with (5.4), Formulas (5.18) and (5.19) completely determine the solution of the problem (5.1) for $\mathbf{C} = 0$; however, they are considerably more complex than Formulas (5.5) and (5.8) which were constructed on the basis of the nontensor matrizant $\phi(t, t_0, \omega^*)$ However, it is more preferable to use the very simple Formulas (5.16), (5.17) in the frozen coordinate system. Some connection evidently exists between the matrizants $\psi_{\cdot j}{}^{i\cdot}(t, t_0; \mathbf{e})$ and $\phi_{\cdot j}{}^{i\cdot}(t, t_0; \omega^*)$, but it will remain unclarified here.

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BIBLIOGRAPHY

1. Rivlin, R.S., Viscoelastic fluids. Rev. frontier fluid dynamics. New York-London-Sydney, Interscience, 1965. pp. 144-170.
2. Sedov, L.I., Introduction to Continuum Mechanics. Fizmatgiz, 1962.
3. Sedov, L.I., Mathematical methods of constructing new models of continua. Usp. Mat. Nauk, Vol. 20, No. 5, 1965.
4. Biot, M.A., Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena. J. Appl. Phys. Vol. 25, No. 11, 1954.
5. Biot, M.A., Variational principles in irreversible thermodynamics with application to viscoelasticity. Phys. Rev., Vol. 97, No. 6, 1955.
6. Kluitenberg, G.A., Thermodynamical theory of elasticity. Physica, deel 28, No. 3, 1962.
7. Kluitenberg, G.A., A note of the thermodynamics of Maxwell bodies, Kelvin bodies (Voigt bodies) and fluids. Physica, deel 28, No. 6, 1962.
8. Kluitenberg, G.A., On rheology and thermodynamics of irreversible processes. Physica, deel 28, No. 11, 1962.
9. Kluitenberg, G.A., On the thermodynamics of viscosity and plasticity. Physica, deel 29, No. 6, 1963.
10. Oldroyd, J.G., On the formulation of rheological equations of state. Proc. Roy. Soc. Ser. A, Vol. 200, No. 1063, 1950.
11. Ziegler, H., Extremal Principles of the Thermodynamics of Irreversible Processes and Continuum Mechanics. Moscow, "Mir," 1966.
12. De Groot, S.R. and Mazur, P., Nonequilibrium Thermodynamics. Moscow, "Mir", 1964.
13. Khazanovich, T.N., Derivation of the equations of linear viscoelasticity, PMM Vol. 28, No. 6, 1964.
14. Pavlov, V.P. and Vinogradov, G.V., Thermal effects in a flow and stopping of anomalously viscous bodies. Dokl. Akad. Nauk SSSR Vol. 125, No. 5, 1959.
15. Kluitenberg, G.A., A unified thermodynamic theory for large deformations in elastic media and in Kelvin (Voigt) media, and for viscous fluid flow. Physica, deel 30, No. 10, 1964.
16. Oldroyd, J.G., Non-Newtonian Flows of Fluids and Solids. Rheology, (Russian translation) IIL, pp. 757-793, 1962.
17. Oldroyd, J.G., Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. Proc. Roy. Soc., Ser. A, Vol. 245, No. 1241, 1958.
18. Goddard, J.D. and Miller, Ch., Nonlinear effects in the rheology of dilute suspensions. J. Fluid Mech., Vol. 28, part 4, 1967.
19. Williams, M.C. and Bird, R.B., Three-constant Oldroyd model for viscoelastic fluids. Phys. Fluids, Vol. 5, No. 9, 1962.
20. Spriggs, T.W., A four-constant model for viscoelastic fluids. Chem. Eng. Sci., Vol. 20, No. 11, 1965.
21. Bland, D., Linear Viscoelasticity Theory. Moscow, "Mir", 1965.

22. Goddard, J.D., and Miller, Ch., An inverse for the Jaumann derivative and some applications to the rheology of viscoelastic fluids. Rheol. Acta, Bild. 5, Heft 3, 1966.
23. Padden, F.J. and De Witt, T.W., Some rheological properties of concentrated polyisobutylene solutions. J. Appl. Phys., Vol. 25, No. 9, 1954.
24. Coleman, B.D., Makrovitz, H., and Noll, W., Viscometric flows of non-Newtonian fluids. Theory and experiment. Springer. Berlin – Heidelberg – New York, 1966.
25. Leonov, A.I., Theory of thixotropy of viscoelastic media with continuous relaxation-time distribution. PMTF, No. 4, 1964.

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